THE METHOD OF BRACKETS IN EXPERIMENTAL MATHEMATICS

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Dedicated to Professor Mourad Ismail on the occasion of his birthday

1. Introduction

The problem of evaluating definite integrals appears in elementary courses. Given a function f and an interval $[a,b] \subset \mathbb{R}$, the task involves expressing

$$\mathcal{I}(f) = \int_a^b f(x)\,dx \tag{1.1}$$

in terms of the (internal) parameters of f. It is surprising that the methods required in solving this problem depend very strongly on subtle forms of the function f. For instance, `Mathematica` gives

$$\int_0^\infty \frac{dx}{e^x+1} = \log 2 \tag{1.2}$$

but it is unable to evaluate

$$\int_0^\infty \frac{dx}{e^x+1+x}. \tag{1.3}$$

The fact that many integrals *cannot be evaluated* is an all-too-familiar experience to both professional mathematicians as well as beginners. Probably the earliest example of such a phenomena comes in the rectification of the ellipse: given $a > b$ and the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the arc length is given by

$$L(a,b) = 4a \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt. \tag{1.4}$$

Here $k = \sqrt{1-b^2/a^2}$ is the eccentricity of the ellipse.

Naturally, the question of whether an integral can be evaluated in *closed form* depends on the type of functions that are allowed in the answer. For example, the integral appearing in (1.4) is the *complete elliptic integral of the second kind*, denoted by $E(k)$. For relevant information the reader may consult [3] and [11].

A source of interesting integrals comes from Feynman diagrams. These are pictorial representations of elementary particle interactions. The reader will find in [10] and [13] more information about this topic.

Date: February 12, 2017.
Key words and phrases. Definite integrals, method of brackets, heuristic rules, Feynman diagrams.

Figure 1 depicts the interaction of three particles corresponding to the three external lines of momenta P_1, P_2 and P_3. In this case, the Schwinger parametrization produces the integral

$$\begin{aligned} G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}} \\ &\times \exp\left(\sum_{j=1}^{3} x_j m_j^2 \right) \exp\left(-\frac{C_{11}P_1^2 + 2C_{12}P_1 \cdot P_2 + C_{22}P_2^2}{x_1+x_2+x_3} \right) dx_1 dx_2 dx_3. \end{aligned}$$

The algorithms in [7] and [8] give the coefficients $C_{i,j}$ as

$$C_{11} = x_1(x_2+x_3),\ C_{12} = x_1x_3,\ C_{22} = x_3(x_1+x_2). \tag{1.5}$$

Conservation of momentum implies $P_3 = P_1 + P_2$, and after replacing the coefficients $C_{i,j}$ into the equation for G, we obtain

$$\begin{aligned} G &= \frac{(-1)^{-D/2}}{\prod_{j=1}^{3}\Gamma(a_j)} \int_0^\infty \int_0^\infty \int_0^\infty x^{a_1-1} x^{a_2-1} x^{a_3-1} \times \\ &\times \frac{\exp\left(x_1m_1^2 + x_2m_2^2 + x_3m_3^2\right) \exp\left(-\frac{x_1x_2P_1^2 + x_2x_3P_2^2 + x_3x_1P_3^2}{x_1+x_2+x_3} \right)}{(x_1+x_2+x_3)^{D/2}} dx_1dx_2dx_3. \end{aligned}$$

To solve the Feynman diagram in Figure 1, one needs to evaluate the integral G as a function of the variables $P_1, P_2 \in \mathbb{R}^4$, the masses m_i, the dimension D and the parameters a_i.

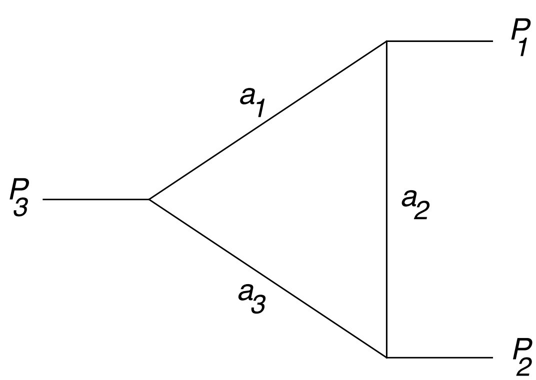

FIGURE 1. The triangle

The special massless case $m_1 = m_2 = m_3 = 0$ has been evaluated in [6] by `the method of brackets` described here. A similar problem, the case of the bubble diagram, is discussed in Example 2.3 below.

The main goal of this note is to introduce this method to a general audience. This is an algorithm for the evaluation of integrals on the half-line $[0, \infty)$ and it consists of a small number of rules. Some of these have been proven, so the method is partly science, while others have been proposed based on the authors' experience. Thus the method falls in the realm of Experimental Mathematics as described in [2].

Still, some rules of this method are in the process of being created, so the method is partly an art.

The method of brackets has provided a powerful and flexible alternative to classical methods of evaluating a class of definite integrals. The reader will find in [5] a selection of entries of the table by Gradshteyn and Ryzhik [9] checked by this method.

2. The algorithm

As advertised in the introductory section, we now reveal an algorithm for the evaluation of definite integrals. The starting point is the notion of *bracket*, defined by the divergent integral

$$\langle a \rangle = \int_0^\infty x^{a-1}\, dx, \quad \text{for } a \in \mathbb{C}, \tag{2.1}$$

and a few set of rules described below.

Rule 1. *Assign to the integral* $I(f) = \int_0^\infty f(x)\, dx$ *a bracket series*

$$\sum_n \phi_n a(n) \langle \alpha n + \beta \rangle. \tag{2.2}$$

Here $\phi_n = \frac{(-1)^n}{n!}$ is called the *indicator* and the coefficients $a(n)$ come from an assumed expansion $f(x) = \sum_{n=0}^{\infty} \phi_n a(n) x^{\alpha n + \beta - 1}$. The extra '$-1$' in the exponent is set for convenience. The coefficients are written as $a(n)$ because these will soon be evaluated at complex numbers n, not necessarily positive integers.

Now we need to state how to convert the bracket series into a number.

Rule 2. *The bracket series* $\sum_n \phi_n a(n) \langle \alpha n + \beta \rangle$ *is assigned the value*

$$\frac{1}{|\alpha|} a(n^*) \Gamma(-n^*). \tag{2.3}$$

Here $\Gamma(x)$ is the Euler's gamma function and n^* is obtained from the vanishing of the brackets; that is, $n^* = -\beta/\alpha$ solves $\alpha n + \beta = 0$. This rule is reminiscent of *Ramanujan's Master Theorem*, for further discussion we refer to [1].

Example 2.1. To compute the integral

$$I_1 = \int_0^\infty e^{-tx}\, dx \tag{2.4}$$

expand the integrand as

$$e^{-tx} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n x^n = \sum_{n=0}^{\infty} \phi_n t^n x^n \tag{2.5}$$

so that $\alpha = \beta = 1$ and $a(n) = t^n$. Then the bracket series is $\sum_n \phi_n t^n \langle n+1 \rangle$ and the evaluation of the integral requires to solve the equation $n + 1 = 0$. Therefore $n^* = -1$ and the integral becomes

$$I_1 = \frac{1}{|1|} t^{-1} \Gamma(1) = \frac{1}{t}. \tag{2.6}$$

And that is all.

The first difficulty in this method comes from the prerequisite of having an explicit form of the coefficients in the expansion. The next example illustrates how to proceed.

Example 2.2. To prove the integral evaluation

$$I_2 = \int_0^\infty e^{-ax} \sin(bx)\, dx = \frac{b}{a^2+b^2}, \tag{2.7}$$

start with the classical expansions

$$e^{-ax} = \sum_{n_1=0}^\infty \phi_{n_1} a^{n_1} x^{n_1} \qquad \text{and} \qquad \sin bx = \sum_{n_2=0}^\infty \phi_{n_2} \frac{n_2!\, b^{2n_2+1}}{(2n_2+1)!} x^{2n_2+1}. \tag{2.8}$$

Replace these in (2.7) to obtain the *two-dimensional bracket series*

$$\sum_{n_1,n_2} \phi_{n_1,n_2} a^{n_1} b^{2n_2+1} \frac{\Gamma(n_2+1)}{\Gamma(2n_2+2)} \langle n_1 + 2n_2 + 2 \rangle \equiv \sum_{n_1,n_2} \phi_{n_1,n_2} C(n_1,n_2) \langle \alpha_{11} n_1 + \alpha_{12} n_2 + \beta_1 \rangle. \tag{2.9}$$

Here $\phi_{n_1,n_2} = \phi_{n_1}\phi_{n_2}$. The relation $n! = \Gamma(n+1)$ has been used to convert factorials in terms of gamma function in anticipation of replacing n by a non-integer value.

Rule 3. *Each representation of an integral by a bracket series has associated an* ***index of the representation*** *according to*

$$\text{index } = \text{number of sums } - \text{ number of brackets}. \tag{2.10}$$

In the case of a multi-dimensional bracket series of positive index, the system generated by the vanishing of the coefficients has a number of free parameters. The solution is then determined upon computing all the contributions of maximal rank in the system by selecting these free parameters. Any two series expressed in the same variable and converging in a common region are added. Divergent series are discarded.

Thus, to evaluate (2.9) proceed as follows: make the brackets vanish and consider two cases treating n_1 and n_2 as free parameters. In the first case, when n_1 is free, the vanishing of the brackets gives $n_2^* = -\frac{\alpha_{11}}{\alpha_{12}} n_1 - \frac{\beta_1}{\alpha_{12}}$. Then the bracket series generates the classical series

$$\frac{1}{|\alpha_{12}|} \sum_{n_1=0}^\infty \phi_{n_1} C(n_1, n_2^*) \Gamma(-n_2^*). \tag{2.11}$$

The case of n_2 is treated in a similar manner.

In the evaluation of (2.7), the case when n_1 is free gives $n_2^* = -\frac{1}{2}n_1 - 1$. Thus one obtains the series

$$T_1 = \frac{1}{2b} \sum_{n_1=0}^\infty \phi_{n_1} \left(\frac{a}{b} \right)^{n_1} \frac{\Gamma(-\frac{n_1}{2})}{\Gamma(-n_1)} \Gamma(\tfrac{n_1}{2}+1). \tag{2.12}$$

To simplify the expression for T_1 observe that the terms with odd n_1 vanish, therefore

$$T_1 = \frac{1}{2b} \sum_{m=0}^\infty \frac{1}{(2m)!} \left(\frac{a}{b} \right)^{2m} \frac{\Gamma(-m)\Gamma(m+1)}{\Gamma(-2m)}. \tag{2.13}$$

The duplication formula for the gamma function in the form

$$\frac{\Gamma(-x)}{\Gamma(-2x)} = \frac{\sqrt{\pi}\, 2^{2x+1}}{\Gamma\left(\frac{1}{2}-x\right)}, \tag{2.14}$$

transforms T_1 into

$$T_1 = \frac{\sqrt{\pi}}{b} \sum_{m=0}^{\infty} \frac{m!}{(2m)!} \frac{1}{\Gamma(\frac{1}{2}-m)} \left(\frac{2a}{b}\right)^{2m} = \frac{1}{b} \sum_{m=0}^{\infty} \left(-\frac{a^2}{b^2}\right)^m, \tag{2.15}$$

after using (see [9, 8.339.3])

$$\Gamma\left(\tfrac{1}{2}-m\right) = \frac{(-1)^m 2^{2m} m! \sqrt{\pi}}{(2m)!}. \tag{2.16}$$

Similarly, the case n_2 free gives $n_1^* = -2n_2 - 2$ and it yields

$$T_2 = \frac{b}{a^2} \sum_{m=0}^{\infty} \left(-\frac{b^2}{a^2}\right)^m. \tag{2.17}$$

The conclusion is that T_1 and T_2 are both given by a series (in this case a geometric series) in the parameters $x = -a^2/b^2$ and $1/x$, respectively. Each one of this series represent the value of (2.7) in complementary regions of convergence.

Example 2.3. This is the evaluation a D-dimensional integral corresponding to the massless bubble Feynman diagram. It is a simpler example than the triangle diagram discussed in the Introduction. The result is well-known [4]. In momentum

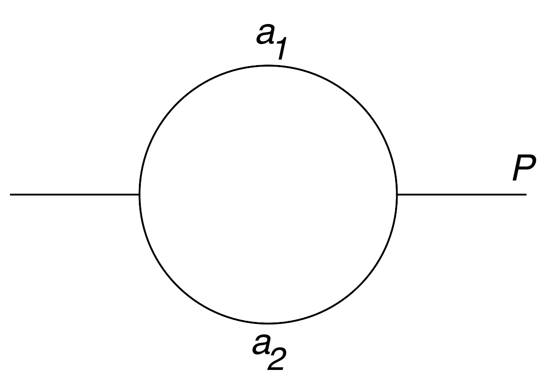

FIGURE 2. The bubble

space the corresponding integral is given by

$$G := \int \frac{1}{i\pi^{D/2}} \frac{1}{[q^2]^{a_1} \left[(p-q)^2\right]^{a_2}}\, d^D q, \tag{2.18}$$

where the parameters $\{a_i\}$ are arbitrary. The Schwinger representation[1] corresponding to this diagram produces

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty x^{a_1-1} y^{a_2-1} \frac{\exp\left(-\frac{xy}{x+y}\, p^2\right)}{(x+y)^{\frac{D}{2}}}\, dx\, dy. \tag{2.19}$$

In order to construct a bracket series for this integral, it is convenient to expand first the exponential function

$$\exp\left(-\frac{xy}{x+y}\, p^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^n},$$

and arrive at

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty x^{a_1-1} y^{a_2-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^{\frac{D}{2}+n}}\, dx\, dy. \tag{2.20}$$

As a next step, apply the binomial expansion to $(x+y)^{-D/2-n}$ so that

$$(x+y)^{-(D/2+n)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\tfrac{D}{2}+n\right)_k x^{-D/2-n-k} y^k \tag{2.21}$$

and replace in (2.20) to obtain, after the change of variables $x \mapsto 1/x$,

$$\begin{aligned} G &= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty \\ &\quad \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} \left(p^2\right)^n \left(\tfrac{D}{2}+n\right)_k x^{k-a_1+\frac{D}{2}} y^{k+n+a_2} \frac{dx}{x} \frac{dy}{y} \\ &= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \phi_{n,k} (p^2)^n \left(\tfrac{D}{2}+n\right)_k \langle k-a_1+\tfrac{D}{2}\rangle\langle k+a_2+n\rangle. \end{aligned}$$

The problem has been reduced to the evaluation of a multi-dimensional bracket series where the number of sums is equal to the number of brackets. This is solved by the next rule.

Rule 4. *Assume the matrix $B=(b_{ij})$ is non-singular, then the assignment is*

$$\sum_{n_1,n_2,\cdots,n_r} \phi_{n_1\cdots n_r} a(n_1,\cdots,n_r)\langle b_{11}n_1+\cdots+b_{1r}n_r+c_1\rangle\cdots\langle b_{r1}n_1+\cdots+b_{rr}n_r+c_r\rangle$$

$$= \frac{1}{|\det(B)|} a(n_1^*,\cdots n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if B is singular.

The reader will now easily verify, in view of Rule 4, that

$$G = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2}-a_1-a_2} \frac{\Gamma(a_1+a_2-\frac{D}{2})\Gamma(\frac{D}{2}-a_1)\Gamma\left(\frac{D}{2}-a_2\right)}{\Gamma(a_1)\Gamma(a_2)\Gamma\left(D-a_1-a_2\right)}. \tag{2.22}$$

[1]There is a canonical procedure associating to each Feynman diagram a multi-dimensional integral. For details, the reader is referred to [12, chapter 3], under the name *alpha parameters.*

A similar procedure evaluates the Feynman diagram for the triangle in Figure 1. The special massless situation: $m_1 = m_2 = m_3 = 0$ and assumption $P_1^2 = P_2^2 = 0$ produces the integral

$$G_1 = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_{\mathbb{R}_+^3} x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\exp\left(-\frac{x_1x_3}{x_1+x_2+x_3}P_3^2\right)}{(x_1+x_2+x_3)^{D/2}}\, dx_1\, dx_2\, dx_3,$$

and the method of brackets then gives

$$\begin{aligned} G_1 &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} (P_3^2)^{D/2-a_1-a_2-a_3} \times \\ &\times \frac{\Gamma(a_1+a_2+a_3-\frac{D}{2})\Gamma(\frac{D}{2}-a_2-a_3)\Gamma(a_2)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-a_1-a_2)}{\Gamma(D-a_1-a_2-a_3)}. \end{aligned}$$

The final example is elementary and is used to illustrate a new rule.

Example 2.4. Entry 3.725.1 in [9] states that

$$\int_0^\infty \frac{\sin ax}{x(x^2+b^2)}\, dx = \frac{\pi}{2b^2}(1-e^{-ab}). \tag{2.23}$$

A (possible) classical evaluation begins by differentiation with respect to a to see that the evaluation is equivalent to

$$\int_0^\infty \frac{\cos ax}{x^2+b^2}\, dx = \frac{\pi e^{-ab}}{2b}. \tag{2.24}$$

Rescaling with the change of variables $x = bt$ shows that it is suffices to prove

$$\int_0^\infty \frac{\cos \alpha t}{t^2+1}\, dt = \frac{\pi}{2}e^{-\alpha}, \tag{2.25}$$

with $\alpha = ab$. The final step is carried out by contour integration.

As a show case, we propose utilizing the method of brackets to evaluate this integral. The first difficult step is come up with a series expansion for the integrand. This task will be simplified by the following instruction.

Rule 5. *For $\alpha \in \mathbb{C}$, the multinomial power $(u_1+u_2+\cdots+u_r)^\alpha$ is assigned the r-dimension bracket series*

$$\sum_{n_1,n_2,\ldots,n_r} \phi_{n_1\,n_2\,\cdots n_r} u_1^{n_1}\cdots u_r^{n_r} \frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}. \tag{2.26}$$

In the current situation,

$$(x^2+b^2)^{-1} \mapsto \sum_{n_1,n_2} \phi_{n_1,n_2} x^{2n_1} b^{2n_2} \langle 1+n_1+n_2\rangle. \tag{2.27}$$

The standard expansion

$$\sin ax = \sum_{n_3=0}^\infty \frac{(-1)^{n_3}}{(2n_3+1)!} a^{2n_3+1}x^{2n_3+1} = \sum_{n_3=0}^\infty \phi_{n_3}\frac{\Gamma(n_3+1)}{\Gamma(2n_3+2)} a^{2n_3+1}x^{2n_3+1},$$

now leads to the bracket series

$$\int_0^\infty \frac{\sin ax\, dx}{x(x^2+b^2)} = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{\Gamma(n_3+1)}{\Gamma(2n_3+2)} a^{2n_3+1} b^{2n_2} \langle n_1+n_2+1\rangle\ \langle 2n_1+2n_3+1\rangle.$$

Rule 3 enforces that the solution is obtained by solving the system

$$\begin{aligned} n_1 + n_2 &= -1 \\ 2n_1 + 2n_3 &= -1. \end{aligned} \tag{2.28}$$

Since the system (2.28) is of rank 1, the solution relies on one free parameter.
Case 1. n_1 is free. Then $n_2 = -1 - n_1$ and $n_3 = -\frac{1}{2} - n_1$ and hence the brackets series is

$$\sum_{n_1=0}^{\infty} \frac{1}{2}(-1)^{n_1}\Gamma(\tfrac{1}{2}+n_1)\frac{\Gamma(\frac{1}{2}-n_1)}{\Gamma(1-2n_1)}a^{-2n_1}b^{-2n_1-2}. \tag{2.29}$$

This series contains a single non-vanishing term, that for $n_1 = 0$. It reduces to $\pi/2b^2$. This is the asymptotic value of the integral as $ab \to \infty$. *This is a typical phenomena.* Series that reduce to a finite number of non-zero terms produce asymptotic expansions of the solution.

Case 2. n_2 is free. Then $n_1 = -1 - n_2$ and $n_3 = \frac{1}{2} + n_2$ and the series becomes

$$\frac{1}{2}\sum_{n_2=0}^{\infty}(-1)^{n_2}\Gamma(-\tfrac{1}{2}-n_2)\frac{\Gamma(\frac{3}{2}+n_2)}{\Gamma(2n_2+3)}a^{2n_2+2}b^{2n_2}. \tag{2.30}$$

Now use $\Gamma(-\frac{1}{2}-n)\Gamma(\frac{1}{2}+n) = (-1)^{n+1}\pi$ to simplify the previous series to

$$-\frac{\pi a^2}{2}\sum_{n_2=0}^{\infty}\frac{(ab)^{2n_2}}{(2n_2+2)!} = \frac{\pi}{2b^2}\left[1-\cosh(ab)\right]. \tag{2.31}$$

Case 3. n_3 is free. Then $n_1 = -\frac{1}{2} - n_3$ and $n_2 = n_3 - \frac{1}{2}$. Then the series becomes

$$\begin{aligned} \frac{a}{2b}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_3}}{\Gamma(2n_3+2)}a^{2n_3}b^{2n_3}\Gamma(n_3+\tfrac{1}{2})\Gamma(\tfrac{1}{2}-n_3) &= \frac{\pi a}{2b}\sum_{n_3=0}^{\infty}\frac{(ab)^{2n_3}}{(2n_3+1)!} \\ &= \frac{\pi}{2b^2}\sinh(ab). \end{aligned}$$

The process yields the asymptotic behavior of the solution and two convergent series. Rule 3 commands that these two convergent series should be added to produce the result. Indeed,

$$\frac{\pi}{2b^2}\left[1-\cosh(ab)\right] + \frac{\pi}{2b^2}\sinh(ab) = \frac{\pi}{2b^2}\left(1-e^{-ab}\right), \tag{2.32}$$

confirms (2.23).

3. Conclusions

The method of brackets provides a flexible procedure to evaluate a large number of definite integrals on the half-line $[0, \infty)$. It consists of a small number of rules to produce, from the integrand, a bracket series and a second set of rules to evaluate these formal series. Some progress has been made in providing rigorous proofs of these rules, but most of them remain in the experimental stage.

4. Acknowledgements

The authors wish to thank T. Ambederhan for improvements on an earlier version of this work.

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