From integrals to multi-sum identities

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Abstract

Ramanujan's Master Theorem and its extension in the method of brackets are a powerful technique for evaluation of many definite integrals, often producing series solutions. In the case that the series are multi-sums, simplification is nontrivial. Recurrence-finding algorithms assist in proving multi-sum identities to verify the correctness of such multi-sum series solutions.

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1. Introduction

Software packages such as Mathematica are often unable to compute integrals of special functions with symbolic parameters. Without an automated method for such problems, one must resort to tables such as [10] when general symbolic solutions are desired. The present work shows how Ramanujan's Master Theorem and its generalization in the *method of brackets* can be combined with recurrence-finding algorithms to solve or verify such integrals with arbitrary integer-valued parameters.

Ramanujan's Master Theorem provides a method of evaluation of a large class of definite integrals using series expansion of the integrand. Proved in [1, 11], Ramanujan's Master Theorem has been extended to the *method of brackets* [7, 9] with the addition of a few heuristic rules. As illustrated in [1, 6], this method evaluates many single or multiple integrals involving special functions. Evaluations of integrals from Feynman diagrams are performed by this method in [8]. Automatization of the method began in [13], and updated code was used in the present work.

For many integrals, this method results in series solutions that must be simplified. Single sum solutions can often be simplified using computer algebra systems, but most multi-sum series cannot be simplified automatically. Beginning with multi-sums from the method of brackets, this work presents and proves identities of the form

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_q=0}^{\infty} F(\mu, n_1, \ldots, n_q) = rhs(\mu), \tag{1}$$

where μ represents one or more integer-valued parameters μ_i (among possibly other parameters).

Section 2 describes the method of solution, which will be semi-automatic, employing recurrence-finding algorithms [14, 18]. With this approach, the problem is reduced to the checking of finitely many initial values, which can be nontrivial. The combination of software and manual simplification illustrates the correctness of the output from the method of brackets, producing multi-sum identities in the process.

Sections 3 and 4 present examples of such double sum and triple sum identities, respectively. These examples illustrate various types of recurrences, some of which can be solved explicitly. Checking of initial conditions is also performed, and this checking may be nontrivial. For some examples, we present alternative automatic or semi-automatic solution methods [12, 14], which depend on the same recurrence-finding

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algorithms used in the present work. Such solutions are presented in the Mathematica notebook, which is available at `http://www.karentkohl.org/papers/Brackets-MultiSum.nb`.

2. Approach

In this work, we study known integrals appearing in the Gradshteyn-Ryzhik table [10], all containing integer-valued symbolic parameters. Mathematica version 11 is unable to solve any of these integrals with symbolic parameters. Most of these integrals can be evaluated using only Ramanujan's Master Theorem, without the need for the additional heuristic rules from the method of brackets. This evaluation produces double or triple sum series for the integrals studied here.

For each multi-sum, we use recurrence-finding algorithms to find a homogeneous recurrence. When the recurrence can be solved explicitly, we do so; for others, we verify that the multi-sum and the known solution from the table satisfy the same recurrence. In either case, initial values of the sum must be computed, finishing the proof of the multi-sum identity. The proofs are valid only for integer values of parameters since initial conditions will be computed or checked only for integer values.

From integrals to multi-sums.

Ramanujan's Master Theorem [1, 11] gives an explicit formula for evaluation of the Mellin transform of a function $f(x)$ in terms of the form of coefficients in the series expansion of $f(x)$.

Theorem 1. *([11], Section 11.2, eq. A) Let $f(x)$ admit a series expansion of the form*

$$f(x) = \sum_{n=0}^{\infty} \lambda(n) \frac{(-x)^n}{n!}$$

in a neighborhood of $x = 0$ with $f(0) = \lambda(0) \neq 0$. Then $\mathcal{M}(f)$, the Mellin transform of $f(x)$, is evaluated by

$$\int_0^\infty x^{s-1} f(x)\, dx = \Gamma(s)\lambda(-s). \tag{2}$$

The *method of brackets* [7, 9] extends Ramanujan's Master Theorem with a few heuristic rules, which were extended in [5, 6, 13]. The method converts an integral into a bracket series through series expansion of integrand factors. Then the bracket series is evaluated by one or more evaluation rules.

Before presenting the rules, we introduce useful notation. For $a \in \mathbb{R}$, the symbol

$$\langle a \rangle \mapsto \int_0^\infty x^{a-1}\, dx \tag{3}$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)} \tag{4}$$

is called the *indicator* associated to the index n. The notation $\phi_{n_1 n_2 \cdots n_r}$ denotes the product $\phi_{n_1}\phi_{n_2}\cdots\phi_{n_r}$.

Rules for the production of bracket series

Rule $\mathbf{P_1}$. If the function f is given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1},$$

then the integral of f over $[0, \infty)$ is converted into a *bracket series* by the procedure

$$\int_0^\infty f(x)\, dx = \sum_{n=0}^{\infty} a_n \langle \alpha n + \beta \rangle.$$

Rule $\mathbf{P_2}$. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \phi_{n_1\, n_2 \cdots n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}.$$

Rule $\mathbf{P_3}$. Each bracket series has associated an *index of the representation* via

$$\text{index } = \text{number of sums } - \text{ number of brackets.}$$

The index is attached to a specific representation of the integral and not just to the integral itself. Among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

Rules for the evaluation of a bracket series

Rule $\mathbf{E_1}$. The one-dimensional bracket series is assigned the value

$$\sum_{n=0}^{\infty} \phi_n f(n) \langle an + b \rangle \mapsto \frac{1}{|a|} f(n^*) \Gamma(-n^*), \tag{5}$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an + b = 0$. This is precisely Ramanujan's Master Theorem (2).

Rule $\mathbf{E_2}$. Assuming the matrix $A = (a_{ij})$ is non-singular, then the assignment for multi-dimensional bracket series of index 0 is

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \phi_{n_1 \cdots n_r} f(n_1, \cdots, n_r) \langle a_{11} n_1 + \cdots + a_{1r} n_r + c_1 \rangle \cdots \langle a_{r1} n_1 + \cdots + a_{rr} n_r + c_r \rangle$$

$$\mapsto \frac{1}{|\det(A)|} f(n_1^*, \cdots n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule $\mathbf{E_3}$. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded. There is no assignment to a bracket series of negative index.

Rule $\mathbf{E_4}$. Let $k \in \mathbb{N}$ be fixed. In the evaluation of series, the rule

$$(-km)_{-m} = \frac{k}{k+1} \frac{(-1)^m (km)!}{((k+1)m)!}$$

must be used to eliminate Pochhammer symbols with negative index and negative integer base.

Proving multi-sum identities.

Wegschaider's algorithm [18], an extension of multivariate WZ summation [19] implemented in the Mathematica package `MultiSum`, computes recurrences for multi-sums. Assuming the summand is proper hypergeometric in all summation variables n_j and in all integer parameters μ_i, Wegschaider's algorithm uses the method of "creative telescoping" to generate a recurrence. The `FindRecurrence` command produces a recurrence for the summand containing polynomial coefficients free of the summation variables in the form

$$\sum_{m \in \mathbb{S}} a_m(\mu) F(\mu + m, n) = \sum_{j=1}^{q} \Delta_{n_j} (r_j(\mu, n) F(\mu, n)), \tag{6}$$

where q is the index defined in Rule P_3 and where Δ_{n_j} denotes the forward difference operator defined by

$$\Delta_{n_j} F(\mu, n) = F(\mu, n_1, n_2, \ldots, n_j + 1, \ldots, n_q) - F(\mu, n).$$

Summing over all $n_j \in \mathbb{N}_0$ produces a recurrence for the multi-sum. In this work, recurrences for all multi-sums will be homogeneous. By rule E_3, the indicators of free summation indices remain, making the terms F vanish outside $\mathbb{N}_0^q$; the summation could instead be viewed as over $\mathbb{Z}^q$. Then the recurrence for the multi-sum will have the form

$$\sum_{n_1,\ldots,n_q=-\infty}^{\infty} \sum_{m\in\mathbb{S}} a_m(\mu) F(\mu + m, n) = \sum_{n_1,\ldots,n_q=-\infty}^{\infty} \sum_{j=1}^{q} \Delta_{n_j}(r_j(\mu, n) F(\mu, n)).$$

The Δ-parts telescope, and the terms F tend toward zero, making the multi-sum satisfy a homogeneous recurrence of the form

$$\sum_{m\in\mathbb{S}} a_m(\mu) SUM(\mu + m, n) = 0. \tag{7}$$

In the `MultiSum` package, the `SumCertificate` command computes a homogeneous recurrence (7) from the recurrence (6) discovered for the summand F. Koutschan's `HolonomicFunctions` package [14] is an alternative Mathematica package for recurrence-finding. Its `FindCreativeTelescoping` command uses a faster method of creative telescoping [15] to produce recurrences, and often these are simpler than those produced by the `MultiSum` package.

Proving multi-sum identities of the form (1) requires showing that both sides of the identity satisfy the same recurrence and that both sides have the same initial values. Given the recurrence output by `SumCertificate`, the `CheckRecurrence` command in `MultiSum` package verifies that the $rhs(\mu)$ expression, unless it involves special functions, satisfies the same recurrence.

In the `HolonomicFunctions` package, the `Annihilator` command will find an operator P for the function F such that $P(F) = 0$. Such operators P will be found even for F involving special functions, making it more robust than the `MultiSum` package.

Once it is known that the multi-sum and solution $rhs(\mu)$ satisfy the same recurrence, initial values must be checked to complete the proof. This checking is done with some visual inspection and manipulation.

3. Double Sum Identities

This section presents integrals of index 2 and proofs of the resulting double sum identities. These examples illustrate variations encountered in this approach, from solvable recurrences to those that can only be verified using known integral results.

Example 1. Entry **7.388.1** reads

$$\int_0^\infty e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(x)\, dx = (-1)^n 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{-\frac{1}{2}\beta^2}, \tag{8}$$

where H_k denotes the Hermite polynomial of degree k. We replace the exponential function and the sine function with their Maclaurin series and the Hermite polynomial with its hypergeometric series representation ([17], eq. 6.41)

$$H_{2n+1}(x) = \frac{(-1)^n (2n+1)!}{n!} 2x\, {}_1F_1\left(\begin{matrix} -n \\ \frac{3}{2} \end{matrix} \,\middle|\, x^2\right) = \frac{(-1)^n (2n+1)!}{n!} 2x \sum_{n_2=0}^{\infty} \frac{(-n)_{n_2} (x^2)^{n_2}}{\left(\frac{3}{2}\right)_{n_2} n_2!}, \tag{9}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. With these replacements and using the definition (4) of the indicator, the integral can be rewritten as

$$\int_0^\infty \left(\sum_{n_3=0}^{\infty} \phi_{n_3} (x^2)^{n_3}\right) \left(\sqrt{2}\beta x \sum_{n_1=0}^{\infty} \phi_{n_1} \frac{(\sqrt{2}\beta x/2)^{2n_1}}{\left(\frac{3}{2}\right)_{n_1}}\right) \left(\frac{(-1)^n (2n+1)!}{n!} 2x \sum_{n_2=0}^{\infty} \phi_{n_2} \frac{(-n)_{n_2} (x^2)^{n_2}}{\left(\frac{3}{2}\right)_{n_2} (-1)^{n_2}}\right) dx.$$

Applying the definition (3) of the bracket, the bracket series is

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\phi_{n_1,n_2,n_3}\frac{(-1)^{n-n_2}2^{-\frac{1}{2}-n_1}\beta^{1+2n_1}\pi(2n+1)!(-n)_{n_2}}{n!\Gamma\left(\frac{3}{2}+n_2\right)\Gamma\left(\frac{3}{2}+n_1\right)}\left\langle 2n_1+2n_2+2n_3+3\right\rangle.$$

With three sums in the integrand and only one bracket, this problem is of index 2. Evaluation of the bracket series proceeds with Rule E_1 (Ramanujan's Master Theorem). Solving the equation $2n_1+2n_2+2n_3+3=0$ gives three options for the fixed variable, and each evaluation by Rule E_1 produces a double sum.

Case 1: With n_2 and n_3 free, $n_1^*=-n_2-n_3-3/2$ and completing Rule E_1 produces S_1. Due to the factor $\Gamma(-n_2-n_3)$ in the denominator, this sum vanishes and does not contribute to the value of the integral.

$$\begin{aligned}S_1&=\frac{1}{|2|}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\phi_{n_2,n_3}\frac{(-1)^{n-n_2}2^{-\frac{1}{2}-n_1^*}\beta^{1+2n_1^*}\pi(2n+1)!(-n)_{n_2}}{n!\Gamma\left(\frac{3}{2}+n_2\right)\Gamma\left(\frac{3}{2}+n_1^*\right)}\Gamma(-n_1^*)\\&=\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-1)^{n+n_3}2^{n_2+n_3}\beta^{-2-2n_2-2n_3}\pi(2n+1)!(-n)_{n_2}\Gamma\left(n_2+n_3+\frac{3}{2}\right)}{n!\Gamma(1+n_2)\Gamma\left(\frac{3}{2}+n_2\right)\Gamma(-n_2-n_3)\Gamma(1+n_3)}.\end{aligned}$$

Case 2: With n_1 and n_3 free, we find $n_2^*=-n_1-n_3-3/2$ and find the double sum S_2 where we have rewritten $(-n)_{n_2^*}$ with gamma functions. With $\Gamma(-n)$ and $\Gamma(-n_1-n_3)$ in the denominator, the sum S_2 vanishes and does not contribute.

$$\begin{aligned}S_2&=\frac{1}{|2|}\sum_{n_1=0}^{\infty}\sum_{n_3=0}^{\infty}\phi_{n_1,n_3}\frac{(-1)^{n-n_2^*}2^{-\frac{1}{2}-n_1}\beta^{1+2n_1}\pi(2n+1)!(-n)_{n_2^*}}{n!\Gamma\left(\frac{3}{2}+n_2^*\right)\Gamma\left(\frac{3}{2}+n_1\right)}\Gamma(-n_2^*)\\&=\sum_{n_1=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-1)^{\frac{3}{2}+n+2n_1+2n_3}2^{-\frac{3}{2}-n_1}\beta^{1+2n_1}\pi(2n+1)!\Gamma\left(-n-\frac{3}{2}-n_1-n_3\right)\Gamma\left(\frac{3}{2}+n_1+n_3\right)}{n!\Gamma(-n)\Gamma(n_1+1)\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(-n_1-n_3)\Gamma(n_3+1)}\end{aligned}$$

Case 3: With n_1 and n_2 free, we find $n_3^*=-n_1-n_2-3/2$. Completing Rule E_1 produces S_3.

$$\begin{aligned}S_3&=\frac{1}{|2|}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\phi_{n_1,n_2}\frac{(-1)^{n-n_2}2^{-\frac{1}{2}-n_1}\beta^{1+2n_1}\pi(2n+1)!(-n)_{n_2}}{n!\Gamma\left(\frac{3}{2}+n_2\right)\Gamma\left(\frac{3}{2}+n_1\right)}\Gamma(-n_3^*)\\&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-1)^{n+n_1}2^{-\frac{3}{2}-n_1}\beta^{1+2n_1}\pi(2n+1)!(-n)_{n_2}\Gamma\left(\frac{3}{2}+n_1+n_2\right)}{n!\Gamma(1+n_1)\Gamma\left(\frac{3}{2}+n_1\right)\Gamma(1+n_2)\Gamma\left(\frac{3}{2}+n_2\right)}\end{aligned}$$

Only S_3 remains as the solution by the method of brackets, and we now prove the identity:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\pi\beta^{2n_1+1}(-1)^{n+n_1}(2n+1)!(-n)_{n_2}\Gamma\left(n_1+n_2+\frac{3}{2}\right)}{2^{n_1+\frac{3}{2}}\Gamma(n+1)\Gamma(n_1+1)\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(n_2+1)\Gamma\left(n_2+\frac{3}{2}\right)}=(-1)^n2^{n-\frac{1}{2}}\pi^{1/2}\beta^{2n+1}e^{-\frac{1}{2}\beta^2}. \qquad (10)$$

We enter this summand into the Mathematica notebook, and search for recurrences in n for the summand and for the double sum. Wegschaider's algorithm finds two recurrences for the double sum:

```
In[1]:= << MultiSum.m
In[2]:= Sumd1[n1_, n2_, n_, β_] := (-1)^(n+n1) 2^(-3/2-n1) β^(1+2n1) π Γ(2+2n) Pochhammer[-n, n2] Γ(3/2+n1+n2)/(Γ(1+
        n)Γ(1 + n1)Γ(3/2 + n1)Γ(1 + n2)Γ(3/2 + n2))
In[3]:= FindRecurrence[Sumd1[n1, n2, n, β], {n}, {n1, n2}];
In[4]:= rec1 = SumCertificate[%]
```

Out[4]= $\{-2\beta^2 SUM[n-1]-SUM[n]=0,$
$2\left(\beta^2+8n^2-4\beta^2n+4n\right)SUM[n]+16\beta^2n(2n+1)SUM[n-1]+(1-4n)SUM[n+1]=0\}$

The second recurrence is a multiple of the first so we continue with the first, which is solved explicitly by

$$SUM(n)=(-2\beta^2)^n\,SUM(0). \qquad (11)$$

To complete the solution, we need only one initial value at $n=0$. With $n=0$ in the double sum, the sum over n_2 is finite due to the Pochhammer of a negative integer, $(-n)_{n_2}$; the only term corresponds to $n_2=0$. By summing first over n_2, the sum becomes a single sum, which simplifies to the right side of (8) at $n=0$:

$$SUM(0) = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}\beta^{2n_1+1}\pi\Gamma\left(\frac{3}{2}+n_1\right)}{2^{n_1+3/2}n_1!\Gamma\left(\frac{3}{2}+n_1\right)0!\Gamma\left(\frac{3}{2}\right)} = \frac{2\sqrt{\pi}}{2^{3/2}}\beta\sum_{n_1=0}^{\infty}\frac{1}{n_1!}\left(-\frac{\beta^2}{2}\right)^{n_1} = \sqrt{\frac{\pi}{2}}\beta e^{-\beta^2/2}$$

This value for $SUM(0)$, together with the recurrence solution (11), produces the solution (12), which is equivalent to the right side of (8) given in the table [10]:

$$SUM(n) = (-2\beta^2)^n\sqrt{\frac{\pi}{2}}\beta e^{-\beta^2/2} \tag{12}$$

This completes the proof of the identity (10). Simplifying this identity using the duplication formula for the gamma function ([16], Section 19, eq. 2) and substituting $\alpha = \beta^2/2$ gives the simpler form

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-n)_{n_2}\Gamma\left(n_1+n_2+\frac{3}{2}\right)(-\alpha)^{n_1}}{\Gamma(n_1+1)\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(n_2+1)\Gamma\left(n_2+\frac{3}{2}\right)} = \frac{\alpha^n e^{-\alpha}}{\Gamma(n+\frac{3}{2})}.$$

Alternative solutions. Creative telescoping for the original integrand and for the integrand of the contour integral from the Mellin transform method produces the same recurrence as was found for the double sum. For both of these methods, initial value checking requires computation of the integral in (8) for $n=0$, and this value can be computed and checked automatically in Mathematica.

Example 2. Entry **6.522.15** gives the result

$$\int_0^\infty x^{\nu+1}J_\nu(bx)K_\nu(ax)J_\nu(cx)\,dx = \frac{2^{3\nu}(abc)^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\left(\ell_2^2-\ell_1^2\right)^{2\nu+1}} \tag{13}$$

where $\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2}-\sqrt{(b-c)^2+a^2}\right]$ and $\ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2}+\sqrt{(b-c)^2+a^2}\right]$.

We use the integral representation of the Bessel-K function (entry 8.432.5 in [10]),

$$K_\nu(x) = \frac{2^\nu\Gamma\left(\frac{1}{2}+\nu\right)}{\sqrt{\pi}x^\nu}\int_0^\infty \frac{\cos(xt)}{(t^2+1)^{\nu+\frac{1}{2}}}\,dt,$$

to rewrite the integral in (13) as the double integral

$$\int_0^\infty\int_0^\infty \frac{2^\nu\Gamma\left(\frac{1}{2}+\nu\right)x^{\nu+1}\cos(axt)J_\nu(bx)J_\nu(cx)}{\sqrt{\pi}x^\nu\,(t^2+1)^{\nu+\frac{1}{2}}}\,dt\,dx.$$

To form the bracket series, we replace the cosine with its series expansion

$$\cos(axt) = \sum_{n_1=0}^{\infty}\phi_{n_1}\frac{(axt)^{2n_1}}{\left(\frac{1}{2}\right)_{n_1}2^{2n_1}}$$

and each Bessel-J function with its hypergeometric representation

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}{}_0F_1\left(\begin{matrix}-\\ \nu+1\end{matrix}\,\middle|\,-\frac{z^2}{4}\right) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(\nu+1)_k k!4^k}.$$

Rewriting the multinomial power by Rule P_2 introduces two sums and one bracket:

$$(t^2+1)^{-\nu-1/2} = \sum_{n_4}\sum_{n_5}\frac{t^{2n_4}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left\langle \nu+\tfrac{1}{2}+n_4+n_5\right\rangle.$$

To complete the bracket series, we apply the definition of the bracket (3) to the integration variables x and t. With five sums and three brackets, the bracket series is of index 2.

$$\sum_{n_1,n_2,n_3,n_4,n_5} \phi_{1,2,3,4,5} \frac{b^{\nu+2n_2} c^{\nu+2n_3} a^{2n_1-\nu}}{\Gamma^2(\nu+1)(\nu+1)_{n_2}(\nu+1)_{n_3} 2^{2n_1+2n_2+2n_3+\nu}\sqrt{\pi}\left(\frac{1}{2}\right)_{n_1}} \times \left\langle \nu+\tfrac{1}{2}+n_4+n_5 \right\rangle \langle 2n_1+2n_2+2n_3+2\nu+2\rangle \langle 2n_1+2n_4+1\rangle$$

We evaluate the bracket series by Rule E_2, where we set up the system of equations

$$\begin{aligned} \nu+\tfrac{1}{2}+n_4+n_5 &= 0 \\ 2n_1+2n_2+2n_3+2\nu+2 &= 0 \\ 2n_1+2n_4+1 &= 0 \end{aligned}$$

and solve for three n_i values, leaving two free.

Case 1: With n_1 and n_2 free, we find $n_3^* = -n_1-n_2-\nu-1$, $n_4^* = -n_1-1/2$, and $n_5^* = n_1-\nu$. The matrix of coefficients of n_3, n_4, and n_5 has a determinant of 4 so Rule E_2 gives

$$\begin{aligned} S_1 &= \frac{1}{|4|}\sum_{n_1}\sum_{n_2}\phi_{n_1}\phi_{n_2}\frac{b^{\nu+2n_2}c^{\nu+2n_3^*}a^{2n_1-\nu}}{\Gamma^2(\nu+1)(\nu+1)_{n_2}(\nu+1)_{n_3^*}2^{2n_1+2n_2+2n_3^*+\nu}\sqrt{\pi}\left(\frac{1}{2}\right)_{n_1}}\Gamma(-n_3^*)\Gamma(-n_4^*)\Gamma(-n_5^*) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_1+n_2}2^{\nu}a^{2n_1-\nu}b^{2n_2+\nu}c^{-2-2n_1-2n_2-\nu}\Gamma(-n_1+\nu)\Gamma(n_1+n_2+\nu+1)}{\Gamma(1+n_1)\Gamma(-n_1-n_2)\Gamma(1+n_2)\Gamma(1+n_2+\nu)} \end{aligned}$$

Cases 2-7: Similar evaluation by Rule E_2 with other choices of free and fixed variables produces additional double sums:

$$\begin{aligned} S_2 &= \sum_{n_1=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_1+n_3}2^{\nu}a^{2n_1-\nu}b^{-2-2n_1-2n_3-\nu}c^{2n_3+\nu}\Gamma(-n_1+\nu)\Gamma(1+n_1+n_3+\nu)}{\Gamma(1+n_1)\Gamma(-n_1-n_3)\Gamma(1+n_3)\Gamma(1+n_3+\nu)} \\ S_3 &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_2+n_3}2^{\nu}a^{-2-2n_2-2n_3-3\nu}b^{2n_2+\nu}c^{2n_3+\nu}\Gamma(1+n_2+n_3+\nu)\Gamma(1+n_2+n_3+2\nu)}{\Gamma(1+n_2)\Gamma(1+n_3)\Gamma(1+n_2+\nu)\Gamma(1+n_3+\nu)} \\ S_4 &= \sum_{n_2=0}^{\infty}\sum_{n_4=0}^{\infty}\frac{(-1)^{n_2+n_4}2^{\nu}a^{-1-2n_4-\nu}b^{2n_2+\nu}c^{-1-2n_2+2n_4-\nu}\Gamma\left(\frac{1}{2}+n_4\right)\Gamma\left(\frac{1}{2}+n_2-n_4+\nu\right)\Gamma\left(\frac{1}{2}+n_4+\nu\right)}{\Gamma(1+n_2)\Gamma(-n_4)\Gamma(1+n_4)\Gamma\left(\frac{1}{2}-n_2+n_4\right)\Gamma(1+n_2+\nu)} \\ S_5 &= \sum_{n_2=0}^{\infty}\sum_{n_5=0}^{\infty}\frac{(-1)^{n_2+n_5}2^{\nu}a^{2n_5+\nu}b^{2n_2+\nu}c^{-2-2n_2-2n_5-3\nu}\Gamma(-n_5-\nu)\Gamma(1+n_2+n_5+2\nu)}{\Gamma(1+n_2)\Gamma(1+n_5)\Gamma(-n_2-n_5-\nu)\Gamma(1+n_2+\nu)} \\ &= \sum_{n_2=0}^{\infty}\sum_{n_5=0}^{\infty}\frac{(-1)^{n_5}2^{\nu}a^{2n_5+\nu}b^{2n_2+\nu}c^{-2-2n_2-2n_5-3\nu}(1+n_5+\nu)_{n_2}\Gamma(1+n_2+n_5+2\nu)}{\Gamma(1+n_2)\Gamma(1+n_5)\Gamma(1+n_2+\nu)} \\ S_6 &= \sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\frac{(-1)^{n_3+n_4}2^{\nu}a^{-1-2n_4-\nu}b^{-1-2n_3+2n_4-\nu}c^{2n_3+\nu}\Gamma\left(\frac{1}{2}+n_4\right)\Gamma\left(\frac{1}{2}+n_3-n_4+\nu\right)\Gamma\left(\frac{1}{2}+n_4+\nu\right)}{\Gamma(1+n_3)\Gamma(-n_4)\Gamma(1+n_4)\Gamma\left(\frac{1}{2}-n_3+n_4\right)\Gamma(1+n_3+\nu)} \\ S_7 &= \sum_{n_3=0}^{\infty}\sum_{n_5=0}^{\infty}\frac{(-1)^{n_3+n_5}2^{\nu}a^{2n_5+\nu}b^{-2-2n_3-2n_5-3\nu}c^{2n_3+\nu}\Gamma(-n_5-\nu)\Gamma(1+n_3+n_5+2\nu)}{\Gamma(1+n_3)\Gamma(1+n_5)\Gamma(-n_3-n_5-\nu)\Gamma(1+n_3+\nu)} \\ &= \sum_{n_3=0}^{\infty}\sum_{n_5=0}^{\infty}\frac{(-1)^{n_5}2^{\nu}a^{2n_5+\nu}b^{-2-2n_3-2n_5-3\nu}c^{2n_3+\nu}(1+n_5+\nu)_{n_3}\Gamma(1+n_3+n_5+2\nu)}{\Gamma(1+n_3)\Gamma(1+n_5)\Gamma(1+n_3+\nu)} \end{aligned}$$

S_4 and S_6 vanish due to the factor of $\Gamma(-n_4)$ in the denominator. S_1 and S_2 fail to converge due to the factors of the form $\frac{\Gamma(-n_1+\nu)}{\Gamma(-n_1-n_j)}$, which will leave the summand indeterminate once n_1 is at least ν.

Only S_3, S_5, and S_7 remain by Rule E_3, but these exhibit symmetry and different regions of convergence. S_3 converges if $|b|,|c|<|a|$, S_5 converges if $|a|,|b|<|c|$, and S_7 converges if $|a|,|c|<|b|$. Due to the symmetry in the summands, we will continue, letting $SUM(\nu)=S_3$, and will prove the identity

$$SUM(\nu)=\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_2+n_3}2^{\nu}b^{2n_2+\nu}c^{2n_3+\nu}\Gamma(1+n_2+n_3+\nu)\Gamma(1+n_2+n_3+2\nu)}{a^{2+2n_2+2n_3+3\nu}\Gamma(1+n_2)\Gamma(1+n_3)\Gamma(1+n_2+\nu)\Gamma(1+n_3+\nu)}=\frac{2^{3\nu}(abc)^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,(\ell_2^2-\ell_1^2)^{2\nu+1}} \tag{14}$$

We find the recurrence for the summand and then the homogeneous recurrence for the double sum:

```
In[5]:= Sumd2a[n2_, n3_, ν_, a_, b_, c_] := ((-1)^(n2+n3) 2^ν a^(-2-2n2-2n3-3ν) b^(2n2+ν) c^(2*n3+ν) Γ(1 + n2 + n3 + ν)Γ(1 +
        n2 + n3 + 2ν))/(Γ(1 + n2)Γ(1 + n3)Γ(1 + n2 + ν)Γ(1 + n3 + ν))
In[6]:= FindRecurrence [Sumd2a [n2, n3, ν, a, b, c] , {ν}, {n2, n3}]
In[7]:= SumCertificate[%]

Out[7]= {-4abc(-1 + 2ν)SUM[ν - 1] + (a^2 + b^2 - 2bc + c^2)(a^2 + b^2 + 2bc + c^2)SUM[ν] == 0}
```

The recurrence has the solution

$$SUM(\nu)=\left(\frac{4abc}{(a^2+(b-c)^2)(a^2+(b+c)^2)}\right)^{\nu}(2\nu-1)!!\,SUM(0), \tag{15}$$

where the double factorial can be rewritten by

$$(2n-1)!!=(2n-1)(2n-3)\cdots 3\cdot 1=\Gamma\left(n+\tfrac{1}{2}\right)2^n\sqrt{\pi}.$$

We need only the initial value, $SUM(0)$, to complete the solution in (15). Here we sum over n_2 first and find a hypergeometric ${}_2F_1$, convergent for $|b|<|a|$:

$$\begin{aligned}SUM(0)&=\frac{1}{a^2}\sum_{n_3=0}^{\infty}\left(-\tfrac{c^2}{a^2}\right)^{n_3}\sum_{n_2=0}^{\infty}\frac{(1+n_3)_{n_2}(1+n_3)_{n_2}}{(1)_{n_2}n_2!}\left(-\tfrac{b^2}{a^2}\right)^{n_2}\\&=\frac{1}{a^2}\sum_{n_3=0}^{\infty}\left(-\tfrac{c^2}{a^2}\right)^{n_3}{}_2F_1\left(\begin{matrix}1+n_3,1+n_3\\1\end{matrix}\,\middle|\,-\frac{b^2}{a^2}\right).\end{aligned} \tag{16}$$

The hypergeometric function can be reduced to a terminating hypergeometric ${}_2F_1$ by Euler's transformation ([2], eq. 2.2.7) and then to a Legendre polynomial by the representation ([16], Section 93, eq. 6)

$$\begin{aligned}{}_2F_1\left(\begin{matrix}1+n_3,1+n_3\\1\end{matrix}\,\middle|\,-\frac{b^2}{a^2}\right)&=\left(1+\tfrac{b^2}{a^2}\right)^{-1-2n_3}{}_2F_1\left(\begin{matrix}-n_3,-n_3\\1\end{matrix}\,\middle|\,-\frac{b^2}{a^2}\right)\\&=\left(1+\tfrac{b^2}{a^2}\right)^{-1-2n_3}\left(1+\tfrac{b^2}{a^2}\right)^{n_3}P_{n_3}\left(\frac{1-b^2/a^2}{1+b^2/a^2}\right),\end{aligned}$$

allowing the sum (16) to be simplified further to

$$SUM(0)=\frac{1}{a^2+b^2}\sum_{n_3=0}^{\infty}\left(-\tfrac{c^2}{a^2+b^2}\right)^{n_3}P_{n_3}\left(\tfrac{a^2-b^2}{a^2+b^2}\right).$$

The generating function for Legendre polynomials ([16], Section 87, eq. 1) simplifies this sum to the expected result of the right side of (13) at $n=0$,

$$SUM(0) = \frac{1}{a^2+b^2} \frac{1}{\sqrt{1 - 2\left(\frac{a^2-b^2}{a^2+b^2}\right)\left(-\frac{c^2}{a^2+b^2}\right) + \left(-\frac{c^2}{a^2+b^2}\right)^2}}$$
$$= (a^4 + 2a^2b^2 + b^4 + 2a^2c^2 - 2b^2c^2 + c^4)^{-1/2} = \frac{1}{\ell_2^2 - \ell_1^2}.$$

Note that summing over n_3 first would have produced the same result and would have required $|c| < |a|$. If both $|b| < |a|$ and $|c| < |a|$, the sum converges. Using this value for $SUM(0)$ in (15) shows that $SUM(n)$ is as expected from the right side of the entry in the table, completing the proof, for integer values of ν, of the double sum identity (14). The identity (14) can be simplified to an equivalent identity

$$\sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{\left(-\frac{b^2}{a^2}\right)^{n_2} \left(-\frac{c^2}{a^2}\right)^{n_3} \Gamma(1+n_2+n_3+\nu)\Gamma(1+n_2+n_3+2\nu)}{\Gamma(1+n_2)\Gamma(1+n_3)\Gamma(1+n_2+\nu)\Gamma(1+n_3+\nu)} = \frac{2^{2\nu} a^{4\nu+2} \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,(\ell_2^2 - \ell_1^2)^{2\nu+1}},$$

and both are proved for integer values of ν. In the same manner, both S_5 and S_7 have also been verified to equal the right side of (13).

Alternative solutions. The differential version of creative telescoping produces the same recurrence as was found for the double sum. However, the value of the integral (13) with $\nu = 0$ value cannot be computed automatically by Mathematica, and a table lookup would be required.

With the Mellin transform approach, $J_\nu(bx)$ and $K_\nu(ax)$ are replaced with their inverse Mellin transform representations (17) and (18)

$$J_\nu(bx) = \frac{1}{4\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{\nu+s}{s}\right)}{\Gamma\left(1+\frac{\nu-s}{s}\right)} \left(\frac{bx}{2}\right)^{-s} ds \tag{17}$$
$$K_\nu(ax) = \frac{1}{8\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \left(\frac{ax}{2}\right)^{-t} \Gamma\left(\frac{t+\nu}{2}\right) \Gamma\left(\frac{t-\nu}{2}\right) dt. \tag{18}$$

Reversing the order of integration requires entry 6.561.14 to evaluate the inner integral as

$$\int_0^\infty x^{\nu+1-s-t} J_\nu(cx)\, dx = \frac{2^{1+\nu-s-t} c^{-2-\nu+s+t} \Gamma\left(1+\nu-\frac{s+t}{2}\right)}{\Gamma\left(\frac{s+t}{2}\right)}.$$

The resulting double contour integral is

$$-\frac{1}{32\pi^2} \int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma\left(\frac{\nu+s}{s}\right)}{\Gamma\left(1+\frac{\nu-s}{s}\right)} \left(\frac{b}{2}\right)^{-s} \left(\frac{a}{2}\right)^{-t} \Gamma\left(\frac{t+\nu}{2}\right) \Gamma\left(\frac{t-\nu}{2}\right) \left(\frac{2^{1+\nu-s-t} c^{-2-\nu+s+t} \Gamma\left(1+\nu-\frac{s+t}{2}\right)}{\Gamma\left(\frac{s+t}{2}\right)}\right) ds\, dt.$$

The recurrence-finding requires splitting by parity of ν so that the integrand is proper hypergeometric in all integer parameters (and in all variables of integration). This necessary case distinction makes this method the least preferred for this problem. In each case, the right side also satisfies the same recurrence. As in the differential creative telescoping approach, the initial values require computation of the integral (13) for $\nu = 0, 1$, but these cannot be computed automatically.

Example 3. Entry **6.522.3** is similar, reading

$$\int_0^\infty x K_0(ax) J_\nu(bx) J_\nu(cx)\, dx = r_1^{-1} r_2^{-1} (r_2 - r_1)^\nu (r_2 + r_1)^{-\nu} = \frac{\ell_1^\nu}{\ell_2^\nu (\ell_2^2 - \ell_1^2)}, \tag{19}$$

where $r_1 = \sqrt{a^2 + (b-c)^2}$, $r_2 = \sqrt{a^2 + (b+c)^2}$, and ℓ_1 and ℓ_2 are as in the last example. The only contributing sum is found to satisfy a 3-term recurrence:

```
In[8]:= Sumd3[n2_, n3_, ν_, a_, b_, c_] := (-1)^(n2+n3) a^(-2-2n2-2n3-2ν) b^(2n2+ν) c^(2n3+ν) Γ(1+n2+n3+ν)^2/(Γ(1+n2)Γ(1+n3)Γ(1 + n2 + ν)Γ(1 + n3 + ν))
In[9]:= FindRecurrence[Sumd3[n2, n3, ν, a, b, c], ν, n2, n3]
In[10]:= SumCertificate[%]
In[11]:= sc3 = ShiftRecurrence[%, {ν, -1}]

Out[11]= {bcSUM[ν - 2] + (-a^2 - b^2 - c^2)SUM[ν - 1] + bcSUM[ν] == 0}
```

The right side of (19) satisfies the same recurrence:

```
In[12]:= r1[a_, b_, c_] := Sqrt[a^2 + (b - c)^2]
In[13]:= r2[a_, b_, c_] := Sqrt[a^2 + (b + c)^2]
In[14]:= RHS3a[ν_, a_, b_, c_] := (r2[a, b, c] - r1[a, b, c])^ν/(r1[a, b, c] * r2[a, b, c] * (r2[a, b, c] + r1[a, b, c])^ν)
In[15]:= CheckRecurrence[sc3, RHS3a[ν, a, b, c]]

Out[15]= {True}
```

To complete the proof, we need to verify two initial conditions. At $\nu = 0$, we find the sum (16) found in **in 6.522.15**, and the right side of (19) at $\nu = 0$ is also the same. At $\nu = 1$, we sum over n_2 first and rewrite the resulting hypergeometric function using Euler's identity:

$$SUM(1) = \sum_{n_3=0}^{\infty} \frac{b(-1)^{n_3} a^{-2n_3-4} c^{2n_3+1} \Gamma(n_3+2)}{\Gamma(n_3+1)} {}_2F_1\left(\left. \begin{matrix} n_3+2, n_3+2 \\ 2 \end{matrix} \right| -\frac{b^2}{a^2} \right)$$

$$= \frac{bc}{a^4} \sum_{n_3=0}^{\infty} \frac{\left(-\frac{c^2}{a^2}\right)^{n_3} \Gamma(n_3+2)}{\Gamma(n_3+1)} \left(1+\frac{b^2}{a^2}\right)^{-2-2n_3} {}_2F_1\left(\left. \begin{matrix} -n_3, -n_3 \\ 2 \end{matrix} \right| -\frac{b^2}{a^2} \right)$$

We then recognize this hypergeometric function as a Jacobi polynomial and continue simplifying using the generating function for Jacobi polynomials ([2], Theorem 6.4.2):

$$SUM(1) = \frac{bc}{a^4} \sum_{n_3=0}^{\infty} \frac{\left(-\frac{c^2}{a^2}\right)^{n_3} \Gamma(n_3+2)}{\Gamma(n_3+1)} \left(1+\frac{b^2}{a^2}\right)^{-2-2n_3} \frac{1}{n_3+1} \left(1+\frac{b^2}{a^2}\right)^{n_3} P_{n_3}^{(1,0)}\left(\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}\right)$$

$$= \frac{bc}{(a^2+b^2)^2} \left(\frac{2}{\sqrt{1+\frac{2(a^2-b^2)c^2}{(a^2+b^2)^2}+\frac{c^4}{(a^2+b^2)^2}} \left(1+\frac{c^2}{a^2+b^2}+\sqrt{1+\frac{2(a^2-b^2)c^2}{(a^2+b^2)^2+\frac{c^4}{(a^2+b^2)^2}}}\right)} \right).$$

As in (16), the double sum converges if both $|b| < |a|$ and $|c| < |a|$. Algebraic simplification shows that this matches the right side of (19) evaluated at $\nu = 1$.

With both initial conditions checked, the proof of the identity is complete for integer values of ν:

$$\sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(-1)^{n_2+n_3} b^{2n_2+\nu} c^{2n_3+\nu} \Gamma(n_2+n_3+\nu+1)^2}{a^{2n_2+2n_3+2\nu+2} n_2! n_3! \Gamma(n_2+\nu+1)\Gamma(n_3+\nu+1)} = r_1^{-1} r_2^{-1} (r_2-r_1)^{\nu} (r_2+r_1)^{-\nu} = \frac{\ell_1^{\nu}}{\ell_2^{\nu}(\ell_2^2-\ell_1^2)}.$$

Example 4. Now we prove identity **7.388.2**, which reads

$$\int_0^{\infty} e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(ax)\, dx = (-1)^n 2^{-1} \pi^{1/2} (a^2-1)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2} H_{2n+1}\left(\frac{a\beta}{\sqrt{2}(a^2-1)^{1/2}}\right). \qquad (20)$$

Note that taking $a = 1$ gives entry **7.388.1**, our Example 1, shown in (8).

The only contributing sum is

$$SUM(n) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(-1)^{n+n_1}\pi 2^{-n_1-\frac{3}{2}} a^{2n_2+1}\beta^{2n_1+1}\Gamma(2n+2)(-n)_{n_2}\Gamma\left(n_1+n_2+\frac{3}{2}\right)}{\Gamma(n+1)\Gamma(n_1+1)\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(n_2+1)\Gamma\left(n_2+\frac{3}{2}\right)}. \tag{21}$$

Two recurrences are discovered for this double sum:

$$32(a-1)^2(a+1)^2(n-1)n(2n-1)(2n+1)SUM(n-2) - 8a^2n(2n+1)(1-a^2-\beta^2-4n+4a^2n)SUM(n-1)$$
$$-2(-3+3a^2-a^2\beta^2+6n-8a^2n+4a^2\beta^2n+12n^2-16a^2n^2)SUM(n)+(1-4n)SUM(n+1)=0 \tag{22}$$

$$8(a-1)^2(a+1)^2(n-1)(2n-1)SUM(n-2)+2(-1+a^2+a^2\beta^2+4n-4a^2n)SUM(n-1)+SUM(n)=0. \tag{23}$$

We will continue with (23) since (22) is a left-multiple of (23), as shown in the notebook. Since `CheckRecurrence` will not work for special functions, we must find a recurrence for the right side. This recurrence may be discovered by the `Annihilator` command from the `HolonomicFunctions` package; we will illustrate that approach in a future example. A manual approach starting from the 3-term recurrence for the Hermite polynomials ([16], Section 104, eq. 7)

$$H_k(z) = 2zH_{k-1}(z) - 2(k-1)H_{k-2}(z) \tag{24}$$

will produce a 3-term recurrence with steps of 2:

$$H_k(z) = (4z^2-4k+6)H_{k-2}(z) - 4(k-2)(k-3)H_{k-4}(z). \tag{25}$$

To verify that the right side of (20) satisfies the recurrence (23), let $k = 2n+1$ and $z = \frac{a\beta}{\sqrt{2}\sqrt{a^2-1}}$ in (25). Multiplying through by $(-1)^n(a^2-1)^{n-1/2}\frac{\sqrt{\pi}}{2}e^{-\beta^2/2}$ verifies that the right side expression of (20) satisfies the recurrence (23).

Having shown that the right side of (20) satisfies the discovered 3-term recurrence, we must verify that they have the same initial conditions. The 3-term recurrence (23) requires the checking of 2 initial conditions corresponding to $n=0$ and $n=1$. If $n=0$ in the double sum (21), only one term remains in the summation over n_2, and the double sum simplifies to the single sum

$$SUM(0) = \frac{a\beta\pi}{\Gamma(3/2)}\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}\beta^{2n_1}}{2^{n_1+3/2}n_1!} = \frac{a\beta\sqrt{\pi}}{\sqrt{2}}e^{-\beta^2/2},$$

which, with $H_1(x) = 2x$, equals the right side of (20) at $n=0$.

When $n=1$, the sum over n_2 is finite, with only the terms corresponding to $n_2=0$ and to $n_2=1$:

$$\begin{aligned}
SUM(1) &= \sum_{n_1=0}^{\infty}\frac{3(-1)^{1+n_1}\beta^{1+2n_1}\pi}{2^{n_1+1/2}n_1!\Gamma\left(\frac{3}{2}+n_1\right)}\left(\frac{a\Gamma\left(\frac{3}{2}+n_1\right)}{0!\Gamma\left(\frac{3}{2}\right)}+\frac{a^3\Gamma\left(\frac{5}{2}+n_1\right)(-1)}{1!\Gamma\left(\frac{5}{2}\right)}\right)\\
&= -\frac{3a\beta\pi}{\sqrt{2}\Gamma\left(\frac{3}{2}\right)}\sum_{n_1=0}^{\infty}\frac{\left(-\frac{\beta^2}{2}\right)^{n_1}}{n_1!} + \frac{3a^3\beta\pi}{\sqrt{2}\Gamma\left(\frac{3}{2}\right)}\sum_{n_1=0}^{\infty}\frac{\left(-\frac{\beta^2}{2}\right)^{n_1}\left(\frac{5}{2}\right)_{n_1}}{n_1!\left(\frac{3}{2}\right)_{n_1}}\\
&= -\frac{3a\beta\pi}{\sqrt{2}\Gamma\left(\frac{3}{2}\right)}e^{-\beta^2/2} + \frac{3a^3\beta\pi}{\sqrt{2}\Gamma\left(\frac{3}{2}\right)}\left[e^{-\beta^2/2}\left(1-\frac{\beta^2}{3}\right)\right]\\
&= -\sqrt{2\pi}a\beta e^{-\beta^2/2}[3-a^2(3-\beta^2)]
\end{aligned} \tag{26}$$

Comparing (26) with the right side of (20) at $n=1$ and using the value $H_3(x) = 8x^3-12x$ completes the verification of the double sum identity

$$SUM(n)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\pi 2^{-n_1-\frac{3}{2}}a^{2n_2+1}\beta^{2n_1+1}(-1)^{n+n_1}\Gamma(2n+2)(-n)_{n_2}\Gamma\left(n_1+n_2+\frac{3}{2}\right)}{\Gamma(n+1)\Gamma(n_1+1)\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(n_2+1)\Gamma\left(n_2+\frac{3}{2}\right)}$$
$$=(-1)^n2^{-1}\pi^{1/2}(a^2-1)^{n+\frac{1}{2}}e^{-\frac{1}{2}\beta^2}H_{2n+1}\left(\frac{a\beta}{\sqrt{2}(a^2-1)^{1/2}}\right). \quad (27)$$

Letting $a=\sqrt{2}$ in (27) and applying the duplication formula for the gamma function gives a double sum representation for odd Hermite polynomials:

$$H_{2n+1}(\beta)=\frac{\beta\Gamma\left(n+\frac{3}{2}\right)}{2^{n+1}e^{-\frac{1}{2}\beta^2}}\sum_{n_1,n_2=0}^{\infty}\frac{2^{n_2-n_1}\beta^{2n_1}(-1)^{n_1}(-n)_{n_2}\Gamma\left(n_1+n_2+\frac{3}{2}\right)}{\Gamma(n_1+1)\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(n_2+1)\Gamma\left(n_2+\frac{3}{2}\right)}.$$

Example 5. Entry **7.414.9** (28) and the resulting double sum were previously studied in [6]. We now evaluate the double sum using the approach of recurrences. This integral, involving associated Laguerre polynomials $L_m^a(x)$, requires a 2-dimensional recurrence since it contains two integer parameters, m and n.

$$\int_0^{\infty}e^{-x}x^{a+b}L_m^a(x)L_n^b(x)\,dx=(-1)^{m+n}(a+b)!\binom{a+m}{n}\binom{b+n}{m}\quad[\mathrm{Re}(a+b)>-1] \quad (28)$$

As shown in [6], the only contributing double sum is

$$SUM(m,n)=\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{\Gamma(a+m+1)\Gamma(b+n+1)(-m)_{n_2}(-n)_{n_3}\Gamma(a+b+n_2+n_3+1)}{\Gamma(m+1)\Gamma(n+1)\Gamma(n_2+1)\Gamma(n_3+1)\Gamma(a+n_2+1)\Gamma(b+n_3+1)}.$$

The `MultiSum` package discovers three 2-dimensional recurrences. With the `HolonomicFunctions` package, we find a simpler set of recurrences: one in the m dimension, one in the n dimension, and an optional diagonal one. These certify the D-finiteness of $SUM(m,n)$, and only the initial value at the origin is needed.

```
In[16]:= << HolonomicFunctions.m
In[17]:= FindCreativeTelescoping[Sumd5[n2, n3, m, n, a, b], S[n2] − 1, S[n3] − 1, S[n], S[m]]//Factor
In[18]:= FindRelation[%[[1]], Support → 1, S[n], S[m], S[m]S[n]]//Factor
```

Out[18]= $\{(1+n)(1+b-m+n)S_n+(a+m-n)(1+b+n),\ (1+m)(1+a+m-n)S_m+(1+a+m)(b-m+n),$
$(1+m)(1+n)S_nS_m-(1+a+m)(1+b+n)\}$

The right side of (28) satisfies the same recurrences:

```
In[19]:= FindRelation[Annihilator[RHS5[m, n, a, b], {S[m], S[n]}], Support → 1, S[n], S[m], S[m]S[n]]//Factor
```

Out[19]= $\{(1+m)(1+a+m-n)S_m+(1+a+m)(b-m+n),\ (1+n)(1+b-m+n)S_n+(a+m-n)(1+b+n),$
$(1+m)(1+n)S_mS_n-(1+a+m)(1+b+n)\}$

Only the initial value at the origin is needed. Both the sum and the right side of (28) equal $\Gamma(1+a+b)$ at the origin, completing the proof of the identity

$$\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{\Gamma(a+m+1)\Gamma(b+n+1)(-m)_{n_2}(-n)_{n_3}\Gamma(a+b+n_2+n_3+1)}{m!\,n!\,n_2!\,n_3!\,\Gamma(a+n_2+1)\Gamma(b+n_3+1)}$$
$$=(-1)^{m+n}(a+b)!\binom{a+m}{n}\binom{b+n}{m}. \quad (29)$$

Alternative solution. With the Mellin transform approach, the `MultiSum` package finds the same recurrence for the integral as it did for the double sum in (29), and this recurrence is just a multiple/combination of those found by the `HolonomicFunctions` package. Initial values for the integrals can be found automatically in Mathematica.

4. Triple Sum Identities

As the the index increases, the recurrence-finding algorithms require more resources of time and memory. For some problems, the `MultiSum` package was unable to produce a recurrence after several hours. In such cases, the `HolonomicFunctions` package was used instead, with a recursive approach to creative telescoping applied for greatest efficiency.

Example 6. We present a correction to entry **7.388.5** found through this work. Evaluation for small values of n confirms the inclusion of the exponential factor $e^{-\beta^2/2}$ (omitted in [10] and in the source [3]).

$$\int_0^\infty e^{-x^2}[H_n(x)]^2\cos(\sqrt{2}\beta x)\,dx = \sqrt{\pi}2^{n-1}n!e^{-\beta^2/2}L_n(\beta^2), \tag{30}$$

where $L_n(x)$ represents the Laguerre polynomial of degree n. The study of this problem is now split by the parity of the degree of the Hermite polynomial.

Case 1: We first study the case of even values of n, letting $n = 2k$. Using the hypergeometric representation (31) for the even degree polynomial ([17], eq. 6.41),

$$H_{2k}(z) = \frac{(-1)^k\Gamma(2k+1)}{\Gamma(k+1)}\,{}_1F_1\left(\left.\begin{matrix}-k\\1/2\end{matrix}\right|z^2\right), \tag{31}$$

the method of brackets produces four triple sums, but only one contributes, and we will prove the identity

$$\sum_{n_1=0}^\infty\sum_{n_2=0}^\infty\sum_{n_3=0}^\infty \frac{(-1)^{n_3}\beta^{2n_3}[(2k)!]^2(-k)_{n_1}(-k)_{n_2}\Gamma\left(n_1+n_2+n_3+\frac{1}{2}\right)}{2^{n_3+1}(k!)^2\left(\frac{1}{2}\right)_{n_1}n_1!\left(\frac{1}{2}\right)_{n_2}n_2!\left(\frac{1}{2}\right)_{n_3}n_3!} = \sqrt{\pi}2^{2k-1}(2k)!e^{-\beta^2/2}L_{2k}(\beta^2). \tag{32}$$

After entering the summand, we search for recurrences

```
In[20]:= FindRecurrence[Sumd6e[n1, n2, n3, k, β], {k}, {n1, n2}, 1]
In[21]:= SumCertificate[%]
In[22]:= ShiftRecurrence[%[[1]], {k, 1}]
```

Out[22]= $64(1+k)^2(1+2k)^2(7-\beta^2+4k)SUM[k] + 4(-50+58\beta^2-15\beta^4+\beta^6-140k+100\beta^2k-12\beta^4k-120k^2+40\beta^2k^2-32k^3)SUM[k+1] + (3-\beta^2+4k)SUM[k+2] == 0$

We now verify that the right side of (30) satisfies the same recurrence. Starting from the recurrence relation for the Laguerre polynomials ([16], Section 121, eq. 8), manual expansion into a recurrence with steps of size two is possible. Instead, we illustrate the search for an annihilating operator for the corrected right side of (30) using the `HolonomicFunctions` package:

```
In[23]:= RHS6[n_, β_] := √π 2^(n-1) n! e^(-β^2/2) LaguerreL[n, β^2]
In[24]:= << HolonomicFunctions.m
In[25]:= Annihilator[RHS6[2k, β], S[k]]//Factor
```

Out[25]= $\{(3+4k-\beta^2)S_k^2 - 4(50+140k+120k^2+32k^3-58\beta^2-100k\beta^2-40k^2\beta^2+15\beta^4+12k\beta^4-\beta^6)S_k + 64(1+k)^2(1+2k)^2(7+4k-\beta^2)\}$

The annihilating operator gives a recurrence equivalent to the shifted recurrence found above for the triple sum. Initial values to be checked correspond to $k = 0$ and $k = 1$. For the case of $k = 0$ ($n = 0$), the only values of n_1 and n_2 for which the summand is nonzero are $n_1 = n_2 = 0$; then we have a single sum over n_3, which evaluates to $\frac{\sqrt{\pi}}{2}e^{-\beta^2/2}$, equaling the right side of equation (30) with $L_0(z) = 1$. If $k = 1$ ($n = 2$), the sums over both n_1 and n_2 are finite, leaving only nonzero terms for $n_1 = 0, 1$ and $n_2 = 0, 1$. The four single sums over n_3 total $2\sqrt{\pi}e^{-\beta^2/2}(\beta^4 - 4\beta^2 + 2) = 4\sqrt{\pi}e^{-\beta^2/2}L_2(\beta^2)$ with $L_2(z) = \frac{1}{2}(2 - 4z + z^2)$. This check completes the proof of the identity (32).

Case 2: We now consider odd values of n, letting $n = 2k+1$. The procedure is similar, using the representation (9) for the odd Hermite polynomials. The triple sum produced is found to satisfy the recurrence

$$\begin{aligned}
&64(1+k)^2(3+2k)^2(9-B^2+4k)SUM(k)+\\
&\quad 4(-154+118B^2-21B^4+B^6-284k+140B^2k-12B^4k-168k^2+40B^2k^2-32k^3)SUM(k+1)\\
&\qquad +(5-B^2+4k)SUM(k+2)=0. \qquad (33)
\end{aligned}$$

The discovered annihilating operator verifies that, with $n = 2k+1$, the right side of (30) also satisfies the recurrence (33). For $k = 0$ ($n = 1$), the triple sum simplifies to $SUM(0) = \sqrt{\pi}e^{-\frac{\beta^2}{2}}\left(-\left(\beta^2-1\right)\right)$, equal to the right side of (30) at $n = 1$ since $L_1(x) = 1 - x$. For $k = 1$ ($n = 3$), the simplified sum is $SUM(1) = -4\sqrt{\pi}e^{-\frac{\beta^2}{2}}\left(\beta^6 - 9\beta^4 + 18\beta^2 - 6\right)$, equal to the right side at $n = 3$ since $L_3(x) = \frac{1}{6}(6 - 18x + 9x^2 - x^3)$, finishing the proof of the identity

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_3}\beta^{2n_3}[(2k+1)!]^2(-k)_{n_1}(-k)_{n_2}\Gamma\left(n_1+n_2+n_3+\frac{3}{2}\right)}{2^{n_3-1}(k!)^2 n_1!\left(\frac{3}{2}\right)_{n_1} n_2!\left(\frac{3}{2}\right)_{n_2}\left(\frac{1}{2}\right)_{n_3} n_3!} = \sqrt{\pi}2^{2k}(2k+1)!e^{-\beta^2/2}L_{2k+1}(\beta^2). \qquad (34)$$

Alternative solution. Like the approach above, the Mellin transform approach would also require the parity distinction. However, with a differential approach to creative telescoping, there is no need for the parity distinction, and a more concise proof of the integral identity is found. Both the integral and the right side of (30) are found to satisfy the recurrence

$$INT(n+2) - 2\left(-\beta^2 + 2n + 3\right)INT(n+1) + 4(n+1)^2 INT(n) = 0.$$

Then two initial values for the integral can be computed automatically in Mathematica, and these agree with initial values for the right side. In this example, such an approach is more efficient for verifying the integral identity.

Example 7. We present a variation of entry **7.375.2** with only even orders and over only the half-line:

$$\int_0^{\infty} e^{-x^2} H_{2k}(x)H_{2m}(x)H_{2n}(x)\,dx = \frac{2^{k+m+n-1}\sqrt{\pi}(2k)!(2m)!(2n)!}{(m+n-k)!(k+m-n)!(k+n-m)!}. \qquad (35)$$

The only contributing triple sum is

$$SUM(k,m,n) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{\pi^{3/2}(2k)!(2m)!(2n)!(-1)^{k+m+n}(-k)_{n_1}(-m)_{n_2}(-n)_{n_3}\Gamma\left(n_1+n_2+n_3+\frac{1}{2}\right)}{2\,k!\,m!\,n!\,\Gamma\left(n_1+\frac{1}{2}\right)\Gamma(n_1+1)\Gamma\left(n_2+\frac{1}{2}\right)\Gamma(n_2+1)\Gamma\left(n_3+\frac{1}{2}\right)\Gamma(n_3+1)}.$$

The `FindRecurrence` command produces a 4-term recurrence which is also satisfied by the right side of (35). Alternatively, the `FindCreativeTelescoping` command produces 3 symmetric recurrences:

$$(k+m-n)(k-m+n)SUM(k,m,n) = -4k(2k-1)(k-1-m-n)SUM(k-1,m,n) \qquad (36)$$
$$(k-m-n)(k+m-n)SUM(k,m,n) = -4m(2m-1)(k-m+1+n)SUM(k,m-1,n) \qquad (37)$$
$$(k-m-n)(k-m+n)SUM(k,m,n) = -4n(2n-1)(k+m-n+1)SUM(k,m,n-1). \qquad (38)$$

The three discovered annihilating operators verify that the right side also satisfies the recurrences above. These recurrences each relate adjacent points within a particular plane. However, when the coefficient of $SUM(k,m,n)$ vanishes, the recurrence cannot be applied.

Within the $k = 0$ plane, the coefficient in equations (37) and (38) vanishes along the diagonal $m = n$. Rather than simplifying the triple sum $SUM(0,n,n)$, we compute a recurrence for the case that $m = n$, substitute $k = 0$, and simplify to find a recurrence along the diagonal:

$$SUM(0,n+1,n+1) - 8(1+n)(1+2n)SUM(0,n,n) = 0. \qquad (39)$$

The initial value at the origin is $SUM(0,0,0) = \sqrt{\pi}/2$, which equals the right side value at the origin. With the recurrences (37), (38), and (39), any value in the $k = 0$ coordinate plane can be found.

In the $m = 0$ plane, the coefficient vanishes for $k = n$, and for $n = 0$, the vanishing happens for $k = m$. Similarly, we find recurrences for the diagonals within the other coordinate planes:

$$SUM(n+1, 0, n+1) = 8(1+n)(1+2n)SUM(n, 0, n) \tag{40}$$

$$SUM(m+1, m+1, 0) = 8(1+m)(1+2m)SUM(m, m, 0). \tag{41}$$

With this set of six recurrences and the initial value $SUM(0,0,0) = \sqrt{\pi}/2$, any value in the octant can be computed.

Example 8. Entry **7.418.5** reads

$$\int_0^\infty xe^{-\frac{1}{2}x^2} L_n^\alpha\left(\tfrac{1}{2}x^2\right) L_n^{\frac{1}{2}-\alpha}\left(\tfrac{1}{2}x^2\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2} ye^{-\frac{1}{2}y^2} L_n^\alpha\left(\tfrac{1}{2}y^2\right) L_n^{\frac{1}{2}-\alpha}\left(\tfrac{1}{2}y^2\right). \tag{42}$$

Of the four triple sums produced by the method of brackets, only the one in (43) contributes.

In this example, commands from `HolonomicFunctions` package were used since the `FindRecurrence` command was unable to produce a result within several hours. Even with the standard form of calling `FindCreativeTelescoping`, over an hour was required, but a recursive application took only a minute to produce a 5-term recurrence with $SUM(n)$ depending on the four immediately preceding terms. Due to its length, we have not reproduced the recurrence here. The `Annihilator` command showed that the right side of (42) satisfied the same 5-term recurrence.

For the initial values of $n = 0, 1, 2, 3$, summation over both n_1 and n_2 is finite. At these values of n, simplification of the triple sum gives the same as in the right side of (42), completing the proof of the identity

$$\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \sum_{n_3=0}^\infty \frac{(-1)^{n_3}\sqrt{\pi}y^{1+2n_3}\Gamma\left(\frac{3}{2}-\alpha+n\right)\Gamma(1+\alpha+n)(-n)_{n_1}(-n)_{n_2}\Gamma\left(\frac{3}{2}+n_1+n_2+n_3\right)}{2^{\frac{1}{2}+n_3}n!^2\,\Gamma(1+n_1)\Gamma\left(\frac{3}{2}-\alpha+n_1\right)\Gamma(1+n_2)\Gamma(1+\alpha+n_2)\Gamma(1+n_3)\Gamma\left(\frac{3}{2}+n_3\right)}$$
$$= \left(\frac{\pi}{2}\right)^{1/2} ye^{-\frac{1}{2}y^2} L_n^\alpha\left(\tfrac{1}{2}y^2\right) L_n^{\frac{1}{2}-\alpha}\left(\tfrac{1}{2}y^2\right). \tag{43}$$

Examples 9 and 10. Entries **7.423.1** and **7.423.2** require corrections in [10] and in the source [4]. The corrected versions are presented here:

$$\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\tfrac{1}{2}x^2\right) H_{2n+1}\left(\tfrac{x}{\sqrt{2}}\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2} L_n\left(\tfrac{1}{2}y^2\right) H_{2n+1}\left(\tfrac{y}{\sqrt{2}}\right) \tag{44}$$

$$\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\tfrac{1}{2}x^2\right) H_{2n}\left(\tfrac{x}{\sqrt{2}}\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2} L_n\left(\tfrac{1}{2}y^2\right) H_{2n}\left(\tfrac{y}{\sqrt{2}}\right). \tag{45}$$

On one hand, they can be viewed as special cases of entries **7.418.5** (42) and **7.418.6** of [10] by relating Hermite polynomials and Laguerre polynomials through the equations below ([2], eqs. 6.2.10 and 6.2.11):

$$H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{-1/2}(x^2)$$
$$H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{1/2}(x^2).$$

On the other hand, the method of brackets produces triple sums for each integral. The triple sums are verified by the recurrence-finding approach to be equivalent to the right sides of (44) and (45). Each problems requires a 4-term recurrence, discovered (and verified for the right sides) by the commands in the `HolonomicFunctions` package. In both problems, three initial values can be checked easily due to the summation being finite in two summation indices.

5. Conclusions

Ramanujan's Master Theorem and its extension in the method of brackets evaluate many definite integrals by producing series solutions. Fully automatic simplification of multi-sum series solutions is not currently possible. We have presented a semi-automatic approach to simplification or verification, illustrated with several typical examples dependent on integer-valued parameters. The result of this work is a collection of multi-sum identities with proofs as well as an approach to produce other such identities. In some cases, we also gave corrections to table entries.

Other approaches to integral verification such as the differential version of creative telescoping and the Mellin transform method combined with creative telescoping are also able to verify integral identities in this paper. However, the most preferable or efficient method can vary from one problem to another, in order to avoid parity distinctions in particular.

Automatic simplification of initial values can be a challenge for all three methods, as seen in various examples. Initial integral values needed for the alternative methods may need to be looked up in a table if software cannot compute them. When initial value multi-sums are finite in at least one summation index, automatic simplification may be possible, but this is not the case in general.

Unlike the other two methods, the Mellin transform approach does not begin in an automatic way since there may exist several options for the choice of factor(s) to be replaced with their inverse Mellin transform representations. These options need to be considered until an inner integral can be evaluated.

The differential approach to creative telescoping is not applicable to integrals that lack integer-valued parameters. Such problems were outside the scope of this work, but the method of brackets will generally provide a solution although symbolic simplification of multi-sum series again may not be possible. Such problems deserve further study, but the production of a solution in some form is an advantage of the method of brackets.

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