An Implementation of the Method of Brackets for Symbolic Integration

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Abstract

In spite of being a classical problem, the current techniques in Mathematica available for Symbolic Integration are not sufficient to evaluate some quartic integrals or some integrals of functions of Mathematical Physics, such as Bessel functions. The *Method of Brackets* [2], [3], a heuristic process appearing in the evaluation of Feynman diagrams, can be used to evaluate symbolically a large class of single or multiple integrals on $[0, \infty)$. The first implementation of the Method of Brackets has been written by the author in the open-source computer algebra system Sage [5]. This implementation allows experimentation with representations of the integrand, which can affect output and efficiency. An algorithm that chooses the best representation of the integrand is in the process of being developed.

1 The Method of Brackets

The Method of Brackets represents an extension of the so-called *Ramanujan Master Theorem* [1]. This connection will develop into a theoretical foundation for the method.

Theorem 1.1 (Ramanujan Master Theorem) *Suppose* F *admits a Taylor expansion of the form* $F(x) = \sum_{k=0}^{\infty} \varphi(k) \frac{(-x)^k}{k!}$ *in a neighborhood of* $x = 0$ *and* $F(0) = \varphi(0) \neq 0$*. Then* $\int_0^\infty x^{\nu-1} F(x) dx = \Gamma(\nu)\varphi(-\nu)$*.*

In the Method of Brackets, the integrand is replaced by a series representation via a sequence of ad-hoc rules. *Brackets* appear in these rules. The preliminary setup converts the integral into a *bracket-series.* Rules for the evaluation of the bracket-series involve solving a system of linear equations. These rules also determine conditions required on parameters for the convergence of the integral.

1.1 Definitions

1. A *bracket* $\langle a \rangle$ is a symbol associated to the divergent integral $\int_0^\infty x^{a-1} dx$.
2. For a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}$, the *bracket-series* $\sum_n a_n \langle \alpha n + \beta \rangle$ is associated with the integral $\int_0^\infty f(x) dx$.
3. The symbol $\phi_n := \frac{(-1)^n}{\Gamma(n+1)}$ is called the *indicator of n.*

1.2 Rules

1. For complex α, the expression $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the bracket-series $$\sum_{n_1,\cdots,n_r} \phi_1\phi_2\cdots\phi_r a_1^{n_1}\cdots a_r^{m_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}.$$
2. The bracket-series $\sum_n \phi_n f(n) \langle an + b \rangle$ is assigned the value $\frac{1}{a} f(n^*)\Gamma(-n^*)$ where n^* solves $an + b = 0$.

   An $r-$dimensional version of this rule also exists:
   $$\sum_{n_1} \cdots \sum_{n_r} \phi_{n_1}\phi_{n_r} f(n_1, \cdots, n_r) \langle a_{11}n_1 + \cdots + a_{1r}n_r + c_1 \rangle \cdots \times \langle a_{r1}n_1 + \cdots + a_{rr}n_r + c_r \rangle$$ is assigned the value $\frac{1}{|\det(A)|} f(n_1^*, \cdots, n_r^*)\Gamma(n_1^*)\cdots\Gamma(n_r^*)$ where A is the matrix of coefficients (a_{ij}) and $\{n_i^*\}$ is the solution of the linear system obtained by the vanishing of the brackets.
3. In the case where the assignment leaves free parameters, any divergent series in these parameters is discarded. In the case that several choices of free indices are available, the series that converge in a common region are added to contribute to the integral.

2 Examples

2.1 A Bessel Function

To evaluate the integral $\int_0^\infty J_\nu(bx)dx$, the series representation

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

of the order ν Bessel function is used so that the integral is associated with the bracket-series $\sum \phi_{n_1} \frac{\left(\frac{b}{2}\right)^\nu \left(\frac{b^2}{2^2}\right)^n}{\Gamma(n+\nu+1)} \langle 2n + \nu + 1 \rangle$. Application of Rule 2 first requires the solution of the equation $2n + \nu + 1 = 0$ for $n^* = \frac{1}{2}(-1-\nu)$. Finally Rule 2 assigns to the bracket-series the value $\frac{1}{2}\left(\frac{\left(\frac{b}{2}\right)^\nu \left(\frac{b^2}{2^2}\right)^{n^*}}{\Gamma(n^*+\nu+1)}\right)\Gamma(-n^*) = \frac{1}{2}\frac{\left(\frac{b}{2}\right)^\nu \left(\frac{b^2}{2^2}\right)^{(-1-\nu)/2}}{\Gamma(\frac{\nu+1}{2})}\Gamma(\frac{\nu+1}{2}) = \frac{1}{b}$.

2.2 A Product of Bessel Functions

The integral $\int_0^\infty J_\mu(ax)J_\nu(bx)dx$ is associated with the bracket-series

$$\sum_{n_1,n_2} \phi_1\phi_2 \frac{a^{2n_1+\mu} b^{2n_2+\nu}}{2^{2n_1+\mu+2n_2+\nu}\Gamma(\mu+1+n_1)\Gamma(\nu+1+n_2)} \langle 2n_1 + 2n_2 + \mu + \nu + 1 \rangle$$

Rule 2 eliminates one bracket with one sum. Since there are two indices here, one will be fixed and the other free. With n_2 as the free index, the integral is assigned the value

$$\sum_{n_2} \phi_{n_2} \frac{1}{2} \left(\frac{a^{2n_1^*+\mu} b^{2n_2+\nu} 2^{-2n_1^*-\mu-2n_2-\nu}}{\Gamma(\mu+1+n_1^*)\Gamma(\nu+1+n_2)} \right) \Gamma(-n_1^*).$$

Substituting $n_1^* = -\frac{1}{2}(2n_2 + \mu + \nu + 1)$ and simplifying gamma functions using the duplication formula allows recognition of this sum as a hypergeometric function :

$$\frac{b^\nu \Gamma\left(\frac{\mu+\nu+1}{2}\right)}{a^{\nu+1}\Gamma(\nu+1)\Gamma\left(\frac{\mu+1-\nu}{2}\right)} {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right) \text{ for } b < a$$

With n_1 as the free index, ν and μ switch and the convergence condition is $a < b$. These results are exactly those given in [4].

2.3 A Quartic Integral

Integral (10.1) in [2] cannot be evaluated in Mathematica, but it can be evaluated with the Method of Brackets. By Rule 1,

$N_{0,4}(a,m) := \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}$ is assigned the bracket-series

$$\sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{(2a)^{n_2}}{\Gamma(m+1)} \langle 4n_1 + 2n_2 + 1 \rangle \langle m + 1 + n_1 + n_2 + n_3 \rangle.$$

With three indices and two brackets, any one of $\{n_1, n_2, n_3\}$ can be chosen as free with the other two fixed, and the 2-dimensional version of Rule 2 will assign a value to each of these sums. The details appear in [2].

3 Implementation (using Sage)

1. Assign of a bracket-series to the integral:
   (a) Wherever possible, recursively replace integrand factors with their series representations by lookups and applications of Rule 1.
   (b) By Definition 2, convert each integral and its corresponding variable of integration into a bracket-series.
2. If there are as many brackets as summations, the system of one or more linear equations is solved in the application of Rule 2 to find the value of the integral.
3. In the case with more summations than brackets, the procedure is as follows:
   (a) Assemble a matrix from the list of brackets and, from its reduced row echelon form, determine all possibilities of free and fixed indices.
   (b) For each of the above choices, solve a system of linear equations corresponding to the fixed indices, producing a series over the free indices.
   (c) Test the series output from the previous step for equivalence.
   (d) If there is only one free index in the series from the previous step, simplify each series and determine convergence conditions in the following cases (to be extended in the future):
      i. The series can be recognized as hypergeometric. Convergence conditions are automatically determined by properties of hypergeometric functions.
      ii. The series can be found to have only finitely many terms, in which case there are no conditions on convergence.
   (e) Determine common regions of convergence and sum the simplified series that converge within each region.

4 Experimentation

4.1 Correct Representations

Experimentation has shown that the only satisfactory representations of the integrand are those that minimize the difference between the number of summations and the number of brackets. Other representations seem to produce only divergent summations. It appears that as much grouping as possible should be performed to obtain satisfactory representations of the integrand. Experimentation with integrals in [3] illustrates that grouping should be performed even when factoring is not possible.

For example, all five integrands below are equivalent and should integrate to $\frac{1}{2}$. However, only the first two representations produce the $\frac{1}{2}$. These both produce bracket-series with the same minimal difference of zero. Other representations, such as the last three below without both $(x+y)$ groupings, produce bracket-series with one more index than brackets. These sums are all divergent; therefore the method gives no solution for these representations.

1. $\int_0^\infty \int_0^\infty \frac{xydxdy}{(xy(x+y)+(x+y))^2} = \frac{1}{2}$ $[\mathtt{Diff} = 4-4 = 0]$
2. $\int_0^\infty \int_0^\infty \frac{xydxdy}{(xy+1)^2(x+y)^2} = \frac{1}{2}$ $[\mathtt{Diff} = 4-4 = 0]$
3. $\int_0^\infty \int_0^\infty \frac{dxdy}{xy(x+y+1/x+1/y)^2} \to$ no solution $[\mathtt{Diff} = 4-3 = 1]$
4. $\int_0^\infty \int_0^\infty \frac{xydxdy}{(x^2y+xy^2+x+y)^2} \to$ no solution $[\mathtt{Diff} = 4-3 = 1]$
5. $\int_0^\infty \int_0^\infty \frac{xydxdy}{(xy(x+y)+x+y)^2} \to$ no solution $[\mathtt{Diff} = 5-4 = 1]$

4.2 Efficient Representations

Experimentation has also shown that several representations may minimize the difference between indices and brackets. For efficiency, the one with the fewest summations must be selected among these.

For either of the following representations, the Method of Brackets produces the correct answer. The second representation is preferred by Sage, but it requires three summations whereas the first representation requires only two summations. The application of Rule 1 in the second representation accounts for the additional index and bracket.

1. $\int_0^\infty x^{s-1}e^{-\beta x^2}e^{-\gamma x}dx$ has 2 indices and 1 bracket
2. $\int_0^\infty x^{s-1}e^{-\beta x^2-\gamma x}dx$ has 3 indices and 2 brackets

This case of the exponential function of a sum has been handled by an explicit rewrite rule to form the first representation, but representations of other functions may also have similar effects on the efficiency.

5 Future Work

1. automatic determination of the best representation of the integrand which (a) minimizes the difference between the number of summations and the number of brackets and (b) is the most efficient, i.e., has fewest number of summations
2. improved simplification of the sums and improved recognition of summations as hypergeometric
3. recognition of particular hypergeometric functions as special functions
4. expanded determination of regions of convergence
5. simplification of multi-sums
6. automatic determination of series representations
7. evaluate as many definite integrals as possible from [4]

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