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Research Article

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Integrals of Frullani type and the method of brackets

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Abstract: The method of brackets is a collection of heuristic rules, some of which have being made rigorous, that provide a flexible, direct method for the evaluation of definite integrals. The present work uses this method to establish classical formulas due to Frullani which provide values of a specific family of integrals. Some generalizations are established.

Keywords: Definite integrals, Frullani integrals, Method of brackets

MSC: 33C67, 81T18

1 Introduction

The integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\left(\frac{b}{a}\right) \tag{1}$$

appears as entry 3.434.2 in [12]. It is one of the simplest examples of the so-called *Frullani integrals*. These are examples of the form

$$S(a,b) = \int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx, \tag{2}$$

and Frullani's theorem states that

$$S(a,b) = [f(0) - f(\infty)] \log\left(\frac{b}{a}\right). \tag{3}$$

The identity (3) holds if, for example, f' is a continuous function and the integral in (3) exists. Other conditions for the validity of this formula are presented in [3, 13, 16]. The reader will find in [1] a systematic study of the Frullani integrals appearing in [12].

The goal of the present work is to use the *method of brackets*, a new procedure for the evaluation of definite integrals, to compute a variety of integrals similar to those in (1). The method itself is described in Section 2. This is based on a small number of *heuristic rules*, some of which have been rigorously established [2, 8]. The point to be stressed here is that the application of the method of brackets is direct and it reduces the evaluation of a definite integral to the solution of a linear system of equations.

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2 The method of brackets

A method to evaluate integrals over the half-line $[0, \infty)$, based on a small number of rules has been developed in [6, 9–11]. This *method of brackets* is described next. The heuristic rules are currently being placed on solid ground [2]. The reader will find in [5, 7, 8] a large collection of evaluations of definite integrals that illustrate the power and flexibility of this method.

For $a \in \mathbb{R}$, the symbol

$$\langle a \rangle = \int_0^\infty x^{a-1}\, dx, \tag{4}$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}, \tag{5}$$

is called the *indicator* associated to the index n. The notation $\phi_{n_1 n_2 \cdots n_r}$, or simply $\phi_{12\cdots r}$, denotes the product $\phi_{n_1}\phi_{n_2}\cdots\phi_{n_r}$.

Rules for the production of bracket series

Rule P$_1$. If the function f is given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}, \tag{6}$$

with $\alpha, \beta \in \mathbb{C}$, then the integral of f over $[0, \infty)$ is converted into a *bracket series* by the procedure

$$\int_0^\infty f(x)\, dx = \sum_n a_n \langle \alpha n + \beta \rangle. \tag{7}$$

Rule P$_2$. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1}\sum_{n_2}\cdots\sum_{n_r} \phi_{n_1 n_2 \cdots n_r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}. \tag{8}$$

Rules for the evaluation of a bracket series

Rule E$_1$. The one-dimensional bracket series is assigned the value

$$\sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*)\Gamma(-n^*), \tag{9}$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an + b = 0$. This is precisely the Ramanujan's Master Theorem.

The next rule provides a value for multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

Rule E$_2$. Assume the matrix $A = (a_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1}\cdots\sum_{n_r} \phi_{n_1\cdots n_r} f(n_1, \cdots, n_r) \langle a_{11}n_1 + \cdots + a_{1r}n_r + c_1 \rangle \cdots \langle a_{r1}n_1 + \cdots + a_{rr}n_r + c_r \rangle$$

$$= \frac{1}{|\det(A)|} f(n_1^*, \cdots n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets.

Rule E$_3$. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded.

3 The formula in one dimension

The goal of this section is to establish Frullani's evaluation (3) by the method of brackets. The notation $\phi_k = (-1)^k/\Gamma(k+1)$ is used in the statement of the next theorem.

Theorem 3.1. *Assume $f(x)$ admits an expansion of the form*

$$f(x) = \sum_{k=0}^{\infty} \phi_k C(k) x^{\alpha k}, \text{ for some } \alpha > 0 \text{ with } C(0) \neq 0 \text{ and } C(0) < \infty. \tag{1}$$

Then,

$$\begin{aligned} S(a,b) &:= \int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx \\ &= \lim_{\varepsilon \to 0} \frac{1}{|\alpha|} \Gamma\left(\frac{\varepsilon}{\alpha}\right) C\left(-\frac{\varepsilon}{\alpha}\right) \left(a^{-\varepsilon} - b^{-\varepsilon}\right) \\ &= C(0) \log\left(\frac{b}{a}\right), \end{aligned} \tag{2}$$

independently of α.

Proof. Introduce an extra parameter and write

$$S(a,b) = \lim_{\varepsilon \to 0} \int_0^\infty \frac{f(ax) - f(bx)}{x^{1-\varepsilon}}\, dx. \tag{3}$$

Then,

$$\begin{aligned} S(a,b) &= \lim_{\varepsilon \to 0} \int_0^\infty \sum_{k=0}^{\infty} \phi_k C(k) \left(a^{\alpha k} - b^{\alpha k}\right) \int_0^\infty x^{\alpha k + \varepsilon - 1}\, dx \\ &= \lim_{\varepsilon \to 0} \sum_k \phi_k C(k) \left(a^{\alpha k} - b^{\alpha k}\right) \langle \alpha k + \varepsilon \rangle. \end{aligned}$$

The method of brackets gives

$$S(a,b) = \lim_{\varepsilon \to 0} \frac{1}{|\alpha|} \Gamma\left(\frac{\varepsilon}{\alpha}\right) C\left(-\frac{\varepsilon}{\alpha}\right) \left(a^{-\varepsilon} - b^{-\varepsilon}\right). \tag{4}$$

The result follows from the expansions $\Gamma(\varepsilon/\alpha) = \alpha/\varepsilon - \gamma + O(\epsilon)$, $C(-\varepsilon/\alpha) = C(0) + O(\varepsilon)$ and $a^{-\varepsilon} - b^{-\varepsilon} = (\log b - \log a)\, \varepsilon + O(\varepsilon^2)$. □

In the examples given below, observe that $C(0) = f(0)$ and that $f(\infty) = 0$ is imposed as a condition on the integrand.

Example 3.2. *Entry* **3.434.2** *of [12] states the value*

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log \frac{b}{a}. \tag{5}$$

This follows directly from (2).

Note 3.3. *The method of brackets gives a direct approach to Frullani style problems if the expansion* (1) *is replaced by the more general one*

$$f(x) = \sum_{k=0}^{\infty} \phi_k C(k) x^{\alpha k + \beta}, \tag{6}$$

with $\beta \neq 0$ and if the function f does not necessarily have a limit at infinity.

Example 3.4. *Consider the evaluation of*

$$I = \int_0^\infty \frac{\sin ax - \sin bx}{x}\, dx, \tag{7}$$

for $a,\ b > 0$. *The integral is evaluated directly as*

$$I = \int_0^\infty \frac{\sin ax}{x}\, dx - \int_0^\infty \frac{\sin bx}{x}\, dx, \tag{8}$$

and since $a,\ b > 0$, *both integrals are* $\pi/2$, *giving* $I = 0$. *The classical version of Frullani theorem does not apply, since* $f(x)$ *does not have a limit as* $x \to \infty$. *Ostrowski [15] shows that in the case* $f(x)$ *is periodic of period* p, *the value* $f(\infty)$ *might be replaced by*

$$\frac{1}{p}\int_0^p f(x)\, dx. \tag{9}$$

In the present case, $f(x) = \sin x$ *has period* 2π *and mean* 0. *This yields the vanishing of the integral.*

The computation of (7) *by the method of brackets begins with the expansion*

$$\sin x = x \cdot {}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix} \middle| -\tfrac{1}{4}x^2 \right). \tag{10}$$

Here

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^\infty \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{11}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$, *is the classical hypergeometric function. The integrand has the series expansion*

$$\sum_{n \geq 0} \phi_n \frac{(a^{2n+1} - b^{2n+1})}{(\frac{3}{2})_n\, 4^n} x^{2n}, \tag{12}$$

that yields

$$I = \sum_n \phi_n \frac{(a^{2n+1} - b^{2n+1})}{(\frac{3}{2})_n\, 4^n} \langle 2n+1 \rangle. \tag{13}$$

The vanishing of the bracket gives $n^* = -1/2$ *and the bracket series vanishes in view of the factor* $a^{2n+1} - b^{2n+1}$.

Example 3.5. *The next example is the evaluation of*

$$I = \int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx = \log\left(\frac{b}{a}\right), \tag{14}$$

for $a,\ b > 0$. *The expansion*

$$\cos x = \sum_{n=0}^\infty \phi_n \frac{n!}{(2n)!} x^{2n}, \tag{15}$$

and $C(n) = \dfrac{n!}{(2n)!} = \dfrac{\Gamma(n+1)}{\Gamma(2n+1)}$ *in* (1). *Then* $C(0) = 1$ *and the integral is* $I = \log\left(\dfrac{b}{a}\right)$, *as claimed.*

Example 3.6. *The integral*

$$I = \int_0^\infty \frac{\tan^{-1}(e^{-ax}) - \tan^{-1}(e^{-bx})}{x}\, dx, \tag{16}$$

is evaluated next. The expansion of the integrand is

$$\tan^{-1}(e^{-t}) = e^{-t} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2} \quad 1 \\ \frac{3}{2} \end{matrix} \middle| -e^{-2t} \right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)\,\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \sum_{k=0}^{\infty}\phi_k (2n+1)^k t^k$$
$$= \sum_{k=0}^{\infty}\phi_k \left[\frac{1}{2}\sum_{n=0}^{\infty}\phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)\,\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}(2n+1)^k\right] t^k.$$

Therefore,

$$C(k) = \frac{1}{2}\sum_{n=0}^{\infty}\phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)\,\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}(2n+1)^k, \tag{17}$$

and from here it follows that

$$C(0) = \frac{1}{2}\sum_{n=0}^{\infty}\phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)\,\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} = \tan^{-1}(1) = \frac{\pi}{4}. \tag{18}$$

Thus, the integral is

$$I = C(0)\log\left(\frac{b}{a}\right) = \frac{\pi}{4}\log\left(\frac{b}{a}\right). \tag{19}$$

4 A first generalization

This section describes examples of Frullani-type integrals that have an expansion of the form

$$f(x) = \sum_{k\geq 0}\phi_k C(k) x^{\alpha k+\beta}, \tag{20}$$

with $\beta \neq 0$.

Theorem 4.1. *Assume $f(x)$ admits an expansion of the form* (20). *Then,*

$$S(a,b) = \int_0^{\infty}\frac{f(ax)-f(bx)}{x}\,dx \tag{21}$$
$$= \lim_{\varepsilon\to 0}\frac{1}{|\alpha|}\Gamma\left(\frac{\beta}{\alpha}+\frac{\varepsilon}{\alpha}\right) C\left(-\frac{\beta}{\alpha}-\frac{\varepsilon}{\alpha}\right)\left[a^{-\varepsilon}-b^{-\varepsilon}\right].$$

Proof. The method of brackets gives

$$S(a,b;\varepsilon) = \int_0^{\infty}\frac{f(ax)-f(bx)}{x^{1-\varepsilon}} \tag{22}$$
$$= \sum_{k\geq 0}\phi_k C(k)\left[a^{\alpha k+\beta}-b^{\alpha k+\beta}\right]\int_0^{\infty} x^{\alpha k+\beta+\epsilon-1}\,dx$$
$$= \sum_{k}\phi_k C(k)\left[a^{\alpha k+\beta}-b^{\alpha k+\beta}\right]\langle \alpha k+\beta+\varepsilon\rangle$$
$$= \frac{1}{|\alpha|}\Gamma(-k)C(k)\left[a^{\alpha k+\beta}-b^{\alpha k+\beta}\right]$$

with $k = -(\beta+\epsilon)/\alpha$ in the last line. The result follows by taking $\epsilon \to 0$. □

Example 4.2. *The integral*

$$\int_0^{\infty}\frac{\tan^{-1} ax - \tan^{-1} bx}{x} = -\frac{\pi}{2}\log\left(\frac{b}{a}\right) \tag{23}$$

appears as entry 4.536.2 in [12]. It is evaluated directly by the classical Frullani theorem. Its evaluation by the method of brackets comes from the expansion

$$\tan^{-1} x = x \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2} & 1 \\ & \frac{3}{2} \end{matrix} \Bigg| -x^2 \right) = \sum_{k \geq 0} \phi_k \frac{\left(\frac{1}{2}\right)_k \, (1)_k}{\left(\frac{3}{2}\right)_k} x^{2k+1}. \tag{24}$$

Therefore, $\alpha = 2, \; \beta = 1$ *and*

$$C(k) = \frac{\Gamma\left(\frac{1}{2}+k\right)\,\Gamma(1+k)}{2\Gamma\left(\frac{3}{2}+k\right)} = \frac{\Gamma(1+k)}{2k+1}. \tag{25}$$

Then

$$\begin{aligned} \int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} &= \lim_{\varepsilon \to 0} \frac{1}{2}\Gamma\left(\frac{1+\varepsilon}{2}\right) C\left(-\frac{1+\varepsilon}{2}\right)\left[a^{-\varepsilon} - b^{-\varepsilon}\right] \\ &= \lim_{\varepsilon \to 0} \frac{1}{2}\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{1-\varepsilon}{2}\right)\frac{\left[a^{-\varepsilon} - b^{-\varepsilon}\right]}{-\varepsilon} \\ &= -\frac{\pi}{2}\log\left(\frac{b}{a}\right). \end{aligned}$$

5 A second class of Frullani type integrals

Let $f_1, \cdots, f_N$ be a family of functions. This section uses the method of brackets to evaluate

$$I = I(f_1, \cdots, f_N) = \int_0^\infty \frac{1}{x} \sum_{k=1}^N f_k(x)\, dx, \tag{1}$$

subject to the condition $\sum_{k=1}^N f_k(0) = 0$, required for convergence.

The functions $\{f_k(x)\}$ are assumed to admit a series representation of the form

$$f_k(x) = \sum_{n=0}^\infty \phi_n C_k(n) x^{\alpha n}, \tag{2}$$

where $\alpha > 0$ is *independent* of k and $C_k(0) \neq 0$. The coefficients C_k are assumed to admit a meromorphic extension from $n \in \mathbb{N}$ to $n \in \mathbb{C}$.

Theorem 5.1. *The integral* I *is given by*

$$I = -\frac{1}{|\alpha|} \sum_{k=1}^N C_k'(0), \tag{3}$$

where

$$C_k'(0) = \frac{dC_k(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0}. \tag{4}$$

Proof. The proof begins with the expansion

$$\frac{f_k(x)}{x^{1-\varepsilon}} = \sum_{n=0}^\infty \phi_n C_k(n) x^{\alpha n - 1 + \varepsilon} \tag{5}$$

and the bracket series for the integral is

$$I = \lim_{\varepsilon \to 0} \sum_n \phi_n \left(\sum_{k=1}^{N} C_k(n) \right) \langle \alpha n + \varepsilon \rangle \tag{6}$$
$$= \lim_{\varepsilon \to 0} \frac{1}{|\alpha|} \Gamma\left(-\frac{\epsilon}{\alpha}\right) \sum_{k=1}^{N} C_k\left(-\frac{\varepsilon}{\alpha}\right).$$

The result follows by letting $\varepsilon \to 0$. □

Example 5.2. *Entry* 3.429 *in [12] states that*

$$I = \int_0^\infty \left[e^{-x} - (1+x)^{-\mu}\right] \frac{dx}{x} = \psi(\mu), \tag{7}$$

where $\mu > 0$ *and* $\psi(x) = \Gamma'(x)/\Gamma(x)$ *is the digamma function. This is one of many integral representation for this basic function. The reader will find a classical proof of this identity in [14]. The method of brackets gives a direct proof.*

The functions appearing in this example are

$$f_1(x) = e^{-x} = \sum_{n=0}^{\infty} \phi_n x^n, \tag{8}$$

and

$$f_2(x) = -(1+x)^{-\mu} = -\sum_{n=0}^{\infty} \phi_n (\mu)_n x^n, \tag{9}$$

where $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$ *is the Pochhammer symbol (this comes directly from the binomial theorem). The condition* $f_1(0) + f_2(0) = 0$ *is satisfied and the coefficients are identified as*

$$C_1(n) = 1 \text{ and } C_2(n) = -(\mu)_n = -\frac{\Gamma(\mu+n)}{\Gamma(\mu)}. \tag{10}$$

Then, $C_1'(0) = 0$ *and* $C_2'(0) = -\dfrac{\Gamma'(\mu)}{\Gamma(\mu)}$. *This gives the evaluation.*

Example 5.3. *The elliptic integrals* $\mathbf{K}(x)$ *and* $\mathbf{E}(x)$ *may be expressed in hypergeometric form as*

$$\mathbf{K}(x) = \frac{\pi}{2} {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} \right| x^2 \right) \text{ and } \mathbf{E}(x) = \frac{\pi}{2} {}_2F_1\left(\left. \begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} \right| x^2 \right) \tag{11}$$

The reader will find information about these integrals in [4, 17].

Theorem 5.1 is now used to establish the value

$$\int_0^\infty \frac{\pi e^{-ax^2} - \mathbf{K}(bx) - \mathbf{E}(cx)}{x} \, dx = \frac{\pi}{2}\left[\log\left(\frac{bc}{a}\right) - \gamma - 4\log 2 + 1\right]. \tag{12}$$

Here $\gamma = -\Gamma'(1)$ *is Euler's constant.*

The first step is to compute series expansions of each of the terms in the integrand. The exponential term is easy:

$$\pi e^{-ax^2} = \pi \sum_{n_1=0}^{\infty} \frac{(-ax^2)^{n_1}}{n_1!} = \sum_{n_1} \phi_{n_1} a^{n_1} x^{2n_1}, \tag{13}$$

and this gives $C_1(n) = a^n$. *For the first elliptic integral,*

$$\mathbf{K}(bx) = \frac{\pi}{2} {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} \right| b^2x^2 \right)$$

$$= \frac{\pi}{2} \sum_{n_2=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n_2} \left(\frac{1}{2}\right)_{n_2}}{(1)_{n_2}\, n_2!} b^{2n_2} x^{2n_2}$$

$$= \sum_{n_2} \phi_{n_2} \frac{\pi}{2} \left(\frac{(-1)^{n_2} b^{2n_2}}{n_2!} \left(\tfrac{1}{2}\right)_{n_2}^2 \right) x^{2n_2}.$$

Therefore,

$$C_2(n) = \frac{\pi}{2} \frac{\cos(\pi n)\Gamma^2(n+\frac{1}{2})}{\Gamma(n+1)} b^{2n}, \tag{14}$$

where the term $(-1)^n$ *has been replaced by* $\cos(\pi n)$. *A similar calculation gives*

$$C_3(n) = \frac{\pi}{4} \frac{\cos(\pi n)\Gamma(n-\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} c^{2n}. \tag{15}$$

A direct calculation gives

$$C_1'(0) = \log a,\; C_2'(0) = -\frac{\gamma}{2} - \log b - \psi\left(\tfrac{1}{2}\right) \text{ and } C_3'(0) = -\frac{\gamma}{2} - \log c - \psi\left(-\tfrac{1}{2}\right).$$

The result now comes from the values

$$\psi\left(\tfrac{1}{2}\right) = -2\log 2 - \gamma \text{ and } \psi\left(-\tfrac{1}{2}\right) = -2\log 2 - \gamma + 2. \tag{16}$$

Example 5.4. *Let* $a, b \in \mathbb{R}$ *with* $a > 0$. *Then*

$$\int_0^\infty \frac{\exp\left(-ax^2\right) - \cos bx}{x}\, dx = \frac{\gamma - \log a + 2\log b}{2}. \tag{17}$$

To apply Theorem 5.1 start with the series

$$f_1(x) = e^{-ax^2} = \sum_n \phi_n a^n x^{2n} \tag{18}$$

and

$$f_2(x) = \cos bx = \sum_n \phi_n \left[\frac{\Gamma(n+1)}{\Gamma(2n+1)} b^{2n} \right] x^{2n}. \tag{19}$$

In both expansions $\alpha = 2$ *and the coefficients are given by*

$$C_1(n) = a^n \text{ and } C_2(n) = \frac{\Gamma(n+1)}{\Gamma(2n+1)} b^{2n}. \tag{20}$$

Then, $C_1'(0) = \log a$ *and* $C_2'(n) = \dfrac{b^{2n}\Gamma(n+1)}{\Gamma(2n+1)} [2\log b + \psi(n+1) - \psi(2n+1)]$ *yield* $C_2'(0) = 2\log b - \psi(1) = 2\log b + \gamma$. *The value* (17) *follows from here.*

Example 5.5. *The next example in this section involves the Bessel function of order* 0

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2}\right)^{2n} \tag{21}$$

and Theorem 5.1 is used to evaluate

$$\int_0^\infty \frac{J_0(x) - \cos ax}{x}\, dx = \log 2a. \tag{22}$$

This appears as entry 6.693.8 *in [12]. The expansions*

$$J_0(x) = \sum_{n=0}^{\infty} \phi_n \frac{1}{n!\, 2^{2n}} x^{2n} \text{ and } \cos ax = \sum_{n=0}^{\infty} \phi_n \frac{n!}{(2n)!} a^{2n} x^{2n}, \tag{23}$$

show $\alpha = 2$ *and*

$$C_1(n) = \frac{1}{\Gamma(n+1)\,2^{2n}} \text{ and } C_2(n) = -\frac{\Gamma(n+1)}{\Gamma(2n+1)}a^{2n}. \tag{24}$$

Differentiation gives

$$C_1'(n) = -\frac{2\ln 2 + \psi(n+1)}{2^{2n}\Gamma(n+1)}, \tag{25}$$

and

$$C_2'(n) = -\frac{a^{2n}\Gamma(n+1)\,(2\log a + \psi(n+1) - 2\psi(2n+1))}{\Gamma(2n+1)}. \tag{26}$$

Then,

$$C_1'(0) = \gamma - 2\log 2 \text{ and } C_2'(0) = -(\gamma + 2\log a), \tag{27}$$

and the result now follows from Theorem 5.1. The reader is invited to use the representation

$$J_0^2(x) = {}_1F_2\left(\left.\begin{matrix} \frac{1}{2} \\ 1 \quad 1 \end{matrix}\right| -x^2\right), \tag{28}$$

to verify the identity

$$\int_0^\infty \frac{J_0^2(x) - \cos x}{x}\,dx = \log 2. \tag{29}$$

Example 5.6. *The final example in this section is*

$$I = \int_0^\infty \frac{J_0^2(x) - e^{-x^2}\cos x}{x}\,dx. \tag{30}$$

The evaluation begins with the expansions

$$J_0(x) = \sum_{k=0}^\infty \phi_k \frac{x^{2k}}{4^k\,\Gamma(k+1)} \text{ and } \cos x = \sum_{k=0}^\infty \phi_k \frac{\sqrt{\pi}}{4^k\,\Gamma\left(k+\frac{1}{2}\right)}. \tag{31}$$

Then,

$$J_0^2(x) = \sum_{k,n}\phi_{k,n}\frac{1}{4^{k+n}\Gamma(k+1)\Gamma(n+1)}x^{2k+2n}, \tag{32}$$

and

$$e^{-x^2}\cos x = \sum_{k,n}\phi_{k,n}\frac{\sqrt{\pi}}{4^k\,\Gamma\left(k+\frac{1}{2}\right)}x^{2k+2n}. \tag{33}$$

Integration yields

$$\begin{aligned} I &= \int_0^\infty \frac{J_0^2(x) - e^{-x^2}\cos x}{x^{1-\varepsilon}}\,dx \\ &= \sum_{k,n}\phi_{k,n}\left[\frac{1}{4^{k+n}\Gamma(k+1)\Gamma(n+1)} - \frac{\sqrt{\pi}}{4^k\,\Gamma\left(k+\frac{1}{2}\right)}\right]\int_0^\infty x^{2k+2n+\varepsilon-1}\,dx \\ &= \sum_{k,n}\phi_{k,n}\left[\frac{1}{4^{k+n}\Gamma(k+1)\Gamma(n+1)} - \frac{\sqrt{\pi}}{4^k\,\Gamma\left(k+\frac{1}{2}\right)}\right]\langle 2k+2n+\varepsilon\rangle. \end{aligned}$$

The method of brackets now gives

$$I = \lim_{\varepsilon\to 0}\frac{1}{2}\sum_{k=0}^\infty \frac{(-1)^k\,\Gamma\left(k+\frac{\varepsilon}{2}\right)}{k!}\left[\frac{1}{2^{-\varepsilon}\Gamma(k+1)\Gamma(1-k-\varepsilon/2)} - \frac{\sqrt{\pi}}{2^{2k}\,\Gamma\left(k+\frac{1}{2}\right)}\right].$$

The term corresponding to $k = 0$ gives

$$\lim_{\varepsilon\to 0} \frac{1}{2}\Gamma\left(\frac{\epsilon}{2}\right)\left[\frac{1}{2^{-\varepsilon}\Gamma\left(1-\frac{\varepsilon}{2}\right)} - 1\right] = \log 2 - \frac{\gamma}{2} \tag{34}$$

and the terms with $k \geq 1$ as $\varepsilon \to 0$ give

$$-\frac{\sqrt{\pi}}{2}\sum_{k=1}^{\infty}\phi_k \frac{\Gamma(k)}{2^{2k}\Gamma\left(k+\frac{1}{2}\right)} = \frac{1}{4}{}_2F_2\left(\begin{matrix}1 & 1\\ \frac{3}{2} & 2\end{matrix}\middle| -\frac{1}{4}\right). \tag{35}$$

Therefore,

$$\int_0^\infty \frac{J_0^2(x) - e^{-x^2}\cos x}{x}\,dx = \frac{1}{4}\left(4\log 2 - 2\gamma + {}_2F_2\left(\begin{matrix}1 & 1\\ \frac{3}{2} & 2\end{matrix}\middle| -\frac{1}{4}\right)\right). \tag{36}$$

No further simplification seems to be possible.

6 A multi-dimensional extension

The method of brackets provides a direct proof of the following multi-dimensional extension of Frullani's theorem.

Theorem 6.1. *Let $a_j,\ b_j \in \mathbb{R}^+$. Assume the function f has an expansion of the form*

$$f(x_1,\cdots,x_n) = \sum_{\ell_1,\ldots,\ell_n=0}^{\infty} \frac{(-1)^{\ell_1}}{\ell_1!}\cdots\frac{(-1)^{\ell_n}}{\ell_n!}C(\ell_1,\cdots,\ell_n)x_1^{\gamma_1}\cdots x_n^{\gamma_n}, \tag{1}$$

where the γ_j are linear functions of the indices given by

$$\begin{aligned} \gamma_1 &= \alpha_{11}\ell_1 + \cdots + \alpha_{1n}\ell_n + \beta_1 \\ \cdots\cdots\cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \gamma_n &= \alpha_{n1}\ell_1 + \cdots + \alpha_{nn}\ell_n + \beta_n. \end{aligned} \tag{2}$$

Then,

$$\begin{aligned} I &= \int_0^\infty \cdots \int_0^\infty \frac{f(b_1x_1,\cdots,b_nx_n) - f(a_1x_1,\cdots,a_nx_n)}{x_1^{1+\rho_1}\cdots x_n^{1+\rho_n}}dx_1\cdots dx_n \\ &= \frac{1}{|\det A|}\lim_{\varepsilon\to 0}\left[b_1^{\rho_1-\varepsilon}\cdots b_n^{\rho_n-\varepsilon} - a_1^{\rho_1-\varepsilon}\cdots a_n^{\rho_n-\varepsilon}\right]\Gamma(-\ell_1^*)\cdots\Gamma(-\ell_n^*)C(\ell_1^*,\cdots,\ell_n^*), \end{aligned}$$

where $A = \left(\alpha_{ij}\right)$ is the matrix of coefficients in (2) *and ℓ_j^*, $1 \leq j \leq n$ is the solution to the linear system*

$$\begin{aligned} \alpha_{11}\ell_1 + \cdots + \alpha_{1n}\ell_n + \beta_1 - \rho_1 + \varepsilon &= 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad &\cdots\cdots \\ \alpha_{n1}\ell_1 + \cdots + \alpha_{nn}\ell_n + \beta_n - \rho_n + \varepsilon &= 0. \end{aligned} \tag{3}$$

Proof. The proof is a direct extension of the one-dimensional case, so it is omitted. □

Example 6.2. *The evaluation of the integral*

$$I = \int_0^\infty\int_0^\infty \frac{e^{-\mu st^2}\cos(ast) - e^{-\mu st^2}\cos(bst)}{\sqrt{s}}\,ds\,dt \tag{4}$$

uses the expansion

$$f(s,t) = e^{-st^2}\cos(st) = \sum_{n_1}\sum_{n_2} \phi_{1,2} \frac{\sqrt{\pi}}{\Gamma\left(n_2+\frac{1}{2}\right)4^{n_2}} s^{n_1+2n_2} t^{2n_1+2n_2}, \tag{5}$$

with parameters $\rho_1 = -\frac{1}{2}$, $\rho_2 = -1$, $b_1 = a^2/\mu$, $b_2 = \mu/a$, $a_1 = b^2/\mu$, $a_2 = \mu/b$. *The solution to the linear system is* $n_1^* = -\frac{1}{2}$ *and* $n_2^* = -\frac{\varepsilon}{2}$ *and* $|\det A| = 2$. *Then*

$$\begin{aligned} I &= \frac{1}{2}\lim_{\varepsilon\to 0}\left[\left(\frac{a^2}{\mu}\right)^{-1/2-\varepsilon}\left(\frac{\mu}{a}\right)^{-1-\varepsilon} - \left(\frac{b^2}{\mu}\right)^{-1/2-\varepsilon}\left(\frac{\mu}{b}\right)^{-1-\varepsilon}\right] \times \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\varepsilon}{2}\right)4^{-\varepsilon/2}} \\ &= \sqrt{\frac{\pi}{\mu}}\lim_{\varepsilon\to 0}\left[\frac{b^\varepsilon - a^\varepsilon}{\varepsilon}\right] \times \frac{\Gamma(1+\varepsilon)\cos\left(\frac{\pi\varepsilon}{2}\right)}{(ab)^\varepsilon} \\ &= \sqrt{\frac{\pi}{\mu}}\log\left(\frac{b}{a}\right). \end{aligned}$$

The double integral (4) *has been evaluated.*

Example 6.3. *The method is now used to evaluate*

$$\int_0^\infty\int_0^\infty \frac{\sin(\mu x y^2)\cos(axy) - \sin(\mu x y^2)\cos(bxy)}{xy} = \frac{\pi}{2}\log\frac{b}{a}. \tag{6}$$

The evaluation begins with the expansion

$$\begin{aligned} f(x,y) &= \sin(xy^2)\cos(xy) \\ &= \left(xy^2\sum_{n_1\ge 0}\phi_{n_1}\frac{\Gamma\left(\frac{3}{2}\right)(xy^2)^{2n_1}}{\Gamma\left(n_1+\frac{3}{2}\right)4^{n_1}}\right)\left(\sum_{n_2\ge 0}\phi_{n_2}\frac{\Gamma\left(\frac{1}{2}\right)(xy)^{2n_2}}{\Gamma\left(n_2+\frac{1}{2}\right)4^{n_2}}\right) \\ &= \sum_{n_1}\sum_{n_2}\phi_{n_1}\phi_{n_2}\frac{\pi}{2\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(n_2+\frac{1}{2}\right)4^{n_1+n_2}} x^{2n_1+2n_2+1} y^{4n_1+2n_2}. \end{aligned}$$

The parameters are $b_1 = a^2/\mu$, $b_2 = \mu/a$, $a_1 = b^2/\mu$, $a_2 = \mu/b$ *and* $\rho_1 = \rho_2 = 0$. *The solution to the linear system is* $n_1^* = -\frac{1}{2}$ *and* $n_2^* = -\frac{\varepsilon}{2}$ *and* $|\det A| = 4$. *Then,*

$$\begin{aligned} I &= \lim_{\varepsilon\to 0}\frac{a^{-\varepsilon} - b^{-\varepsilon}}{4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\frac{\pi}{2\Gamma(1)\Gamma\left(\frac{1-\varepsilon}{2}\right)4^{-\varepsilon-1)/2}} \\ &= \lim_{\varepsilon\to 0}\frac{\pi^{3/2}4^{\varepsilon/2}}{4}\frac{b^\varepsilon - a^\varepsilon}{(ab)^\varepsilon}\frac{2^{1-2\varepsilon}\sqrt{\pi}\,\Gamma(\varepsilon)}{\pi\, csc\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \\ &= \frac{\pi}{2}\log\left(\frac{b}{a}\right), \end{aligned}$$

as claimed.

7 Conclusions

The method of brackets consists of a small number of heuristic rules that reduce the evaluation of a definite integral to the solution of a linear system of equations. The method has been used to establish a classical theorem of Frullani and to evaluate, in an algorithmic manner, a variety of integrals of *Frullani type*. The flexibility of the method yields a direct and simple solution to these evaluations.

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